Interval observers design for uncertain systems

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Outline

1. Introduction

2. Prediction/Correction approach

3. Interval observers
   - LTI systems
   - Partial linear systems

4. Relaxation of cooperativity - changes of coordinates
   - Time-varying systems
   - Time-varying changes of coordinates of LTV systems
   - Linear Parameter-Varying systems
1 Introduction

2 Prediction/Correction approach

3 Interval observers

4 Relaxation of cooperativity - changes of coordinates
Observation/Control

✓ Linear systems: several constructive results ⇒ freq. approaches, output/state feedback,…

✓ Nonlinear systems: the solutions depend on the nonlinearity structure ⇒
Lipschitzian systems: $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$
⇒ linear approaches can be used to build observers/controllers.

✓ LPV systems (Linear Parameter-Varying): intermediate class between Linear and Nonlinear systems
  - Several techniques allow one to transform/approximate NL into LPV systems
    $\dot{x} = f(x, u) \Rightarrow \dot{x} = A(\theta(t))x + B(\theta(t))u$
  - The nonlinear trajectory belongs into the LPV ones
  - NL ≡ Linear + parameter uncertainties ($\theta(t)$)
Estimation & Uncertainties

Several cases may be met

- Models **without** uncertainties
- Models with uncertain parameters (**constant** or **varying** uncertain parameters)
- Uncertain parameters & unknown inputs

Observers structures

- \( \dot{x} = Ax + Bu; y = Cx \)
  \[ \Rightarrow \text{Luenberger Obs. } \dot{z} = Az + Bu + L(y - Cz). \]
- \( \dot{x} = A(\theta)x + B(\theta)u; y = C(\theta)x \Rightarrow \theta \text{ is known or unknown?} \)

Possible solutions

- Adaptive approaches \( \Rightarrow \) joint estimation of \( x \) and \( \theta \).
- Robust approaches \( \dot{z} = A_\alpha z + B_\alpha u + L(y - C_\alpha z) \) (for some average values \( A_\alpha, B_\alpha \) and \( C_\alpha \)).
- **Set-membership estimation / Interval observers.**
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- \( \dot{x} = A(\theta)x + B(\theta)u; \ y = C(\theta)x \) \( \Rightarrow \) \( \theta \) is known or unknown?

Possible solutions

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- \[ \dot{x} = Ax + Bu; \ y = Cx \]
  \[ \Rightarrow \] Luenberger Obs. \[ \dot{z} = Az + Bu + L(y - Cz). \]
- \[ \dot{x} = A(\theta)x + B(\theta)u; \ y = C(\theta)x \Rightarrow \theta \text{ is known or unknown?} \]

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- **Set-membership estimation / Interval observers.**
Set-membership estimation 1/2

- Without uncertainties $\Rightarrow$ point estimation.
- Systems subject to bounded uncertainties $\Rightarrow$ estimation of a feasible solution set.

$\dot{x} = Ax + Bu$

$\dot{x} = A(\theta)x + B(\theta)u$

✓ Prediction/correction approach


✓ Interval Observers


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Set-membership estimation 1/2

- Without uncertainties $\Rightarrow$ point estimation.
- Systems subject to bounded uncertainties $\Rightarrow$ estimation of a feasible solution set.

✓ Prediction/correction approach
  Jaulin, 2002 ; Kieffer & Walter, 2004 ; Raïssi, Ramdani, Candau, 2004 ...

✓ Interval Observers
Set-membership estimation 1/2

- Without uncertainties $\Rightarrow$ point estimation.
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✓ Prediction/correction approach

✓ Interval Observers
Set-membership estimation 2/2

Given a system described by

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{align*}
\]  
(1)

Definition

The dynamical system

\[
\begin{align*}
\dot{z} &= \alpha(z, y, u) \\
[z^T, \overline{x}^T]^T &= \beta(z, y, u)
\end{align*}
\]  
(2)

is an interval observer for (1) if:

\[
x(0) \leq \underline{x}(0) \leq \overline{x}(0) \quad \Rightarrow \quad -\infty < \underline{x}(t) \leq x(t) \leq \overline{x}(t) < \infty, \quad \forall t \geq 0.
\]  
(3)

Roughly speaking, an interval observer should verify two conditions:

- Inclusion: $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, $\forall t \geq t_0$
- Stability of $\epsilon = x - \underline{x}$ and $\overline{\epsilon} = \overline{x} - x$
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Prediction/Correction - discrete-time systems

Given a system described by:

\[
\begin{cases}
  x(j + 1) = f(x_j, u_j, \theta) \\
  y(j) = g(x_j, u_j, \theta)
\end{cases}
\]  

(4)

At the time instant \( j \), the state \( x_j \) belongs into a set \( X_j \) computed by the recursive algorithm:

✓ Initialization Build a set \( X_0 \) containing \( x_0 \) (may be large)
✓ Prediction \( X_{j\mid j-1} = f(X_{j-1}, u_{j-1}, [\theta]) \)
✓ Correction \( X_j = X_{j\mid j-1} \cap g^{-1}([y_m - \bar{e}, y_m + \bar{e}]) \)

Remark

- For linear systems, the set \( X_j \) could be approximated by ellipsoids, zonotopes, polytopes . . .
- Usually intervals and sub-pavings are used to approximate \( X_j \) for nonlinear systems.
Prediction/Correction - discrete-time systems

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- **Initialization**  
  Build a set \(\mathcal{X}_0\) containing \(x_0\) (may be large)

- **Prediction**  
  \(\mathcal{X}_{j|j-1} = f(\mathcal{X}_{j-1}, u_{j-1}, [\theta])\)

- **Correction**  
  \(\mathcal{X}_j = \mathcal{X}_{j|j-1} \cap g^{-1}([y_m - \bar{e}, y_m + \bar{e}])\)

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  Build a set \( \mathcal{X}_0 \) containing \( x_0 \) (may be large)

- **Prediction**  
  \( \mathcal{X}_{j|j-1} = f(\mathcal{X}_{j-1}, u_{j-1}, [\theta]) \)

- **Correction**  
  \( \mathcal{X}_j = \mathcal{X}_{j|j-1} \cap g^{-1}([y_m - \bar{e}, y_m + \bar{e}]) \)

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- **Initialization** Build a set \( X_0 \) containing \( x_0 \) (may be large)
- **Prediction** \( X_{j|j-1} = f(X_{j-1}, u_{j-1}, [\theta]) \)
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**Remark**

- For linear systems, the set \( X_j \) could be approximated by ellipsoids, zonotopes, polytopes . . .
- Usually intervals and sub-pavings are used to approximate \( X_j \) for nonlinear systems.
Prediction/Correction - illustration

Figure: Prediction/Correction mechanism
Prediction/Correction approach

Interval observers

Relaxation of cooperativity - changes of coordinates

Prediction/Correction - continuous-time systems 1/2

\[
\begin{aligned}
\dot{x} &= f(x, u, \theta) \\
y &= g(x, u, \theta)
\end{aligned}
\]  

✓ Initialization  
Build a set \( X_0 \) containing \( x_0 \)

✓ Prediction  
\( X_{j|j-1} := VODE(X_{j-1}, u(t), [\theta]) \)

✓ Correction  
\( X_j := X_{j|j-1} \cap g^{-1}([y_m - \bar{e}, y_m + \bar{e}]) \)

Prediction: Solve the ODE \( \dot{x} = f(x, u, \theta) \) at each time instant \( t_j \) starting from \( X_{j-1} \) at \( t_{j-1} \).
⇒ Validated integration of uncertain ODEs (interval Taylor expansions, ...)

Correction: Use the posterior information (measurements) available at \( t_j \) to contract (reduce) \( X_{j|j-1} \) ⇒ A Constraint Satisfaction Problem (CSP) which can be solved using interval arithmetics.
Monotone systems: When the system (1) is monotone (i.e. trajectories must preserve a partial ordering on states\(^1\)), the prediction is performed by solving only two deterministic ODEs ⇒ the pessimism due to interval arithmetics can be cancelled.

Non-monotone systems: Several works based on the Müller Theorem or differential inclusions allow one to build suitable dynamical systems:

\[
\begin{align*}
\dot{x} &= f(x, \bar{x}, u, \theta, \bar{\theta}) \\
\dot{\bar{x}} &= \bar{f}(x, \bar{x}, u, \theta, \bar{\theta}) \\
x_0 &= x_0 \\
\bar{x}_0 &= \bar{x}_0
\end{align*}
\] (6)

Remarks

- Rules proposed in (Kieffer & Walter, 2006; Ramdani, Meslem & Candau 2006) allow one to compute some vector fields \(\underline{f}, \bar{f}\).
- The stability of (6) cannot be ensured even if the uncertain system (5) is stable ⇒ this phenomenon is due the coupling \(\underline{x}, \bar{x}\).

\(^1\)Partial ordering on states: \(x_1(t_0) \leq x_2(t_0) \Rightarrow x_1(t) \leq x_2(t), \forall t \geq t_0\)
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   - LTI systems
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Cooperative systems - nonnegative systems

- Given two vectors \( x_1, x_2 \) and two matrices \( A_1, A_2 \), the relations \( x_1 \leq x_2, x_1 \geq x_2, A_1 \leq A_2, A_1 \geq A_2 \) should be understood elementwise.

- A positive semi-definite matrix is denoted by \( P : P = P^T \succeq 0 \).

- A square matrix \( S \in \mathbb{R}^{n \times n} \) is called Metzler if \( S_{i,j} \geq 0, \forall \ 1 \leq i \neq j \leq n \). The set of all Metzler matrices is denoted by \( \mathcal{M} \).

**Theorem**

*Given a Metzler matrix \( S (S \in \mathcal{M}) \), the system*

\[
\dot{z} = Sz + r(t); \quad z \in \mathbb{R}^n; \quad r : \mathbb{R}_+ \to \mathbb{R}_+^n
\]

*is called cooperative or nonnegative. Its trajectories verify:*

\[
z(0) \geq 0 \Rightarrow z(t) \geq 0, \forall t \geq 0.
\]
Case of nonlinearities on the outputs 1/6

Given a system described by:

\[
\begin{aligned}
    \dot{x} &= Ax(t) + \varphi(y(t), \theta, u(t)) \\
    y &= Cx(t)
\end{aligned}
\]  

(7)

- **Assumption 1:** There exists a gain \( L \) such that \((A - LC)\) is Hurwitz and Metzler, i.e. \((A - LC) \in \mathcal{H} & \mathcal{M}\).

- **Assumption 2:** There exist two bracketing functions \( \underline{\varphi}, \overline{\varphi} \) and a positive real vector \( M < +\infty \) such that:

\[
\begin{aligned}
    \underline{\varphi}(y(t), \theta_m, \theta_M, u(t)) &\leq \varphi(y(t), \theta, u(t)) \leq \overline{\varphi}(y(t), \theta_m, \theta_M, u(t)) \\
    \|\overline{\varphi}(y(t), \theta_m, \theta_M, u(t)) - \varphi(y(t), \theta_m, \theta_M, u(t))\| &\leq M \\
    \forall (x, u, \theta) &\in \mathcal{D} \times \mathcal{U} \times [\theta, \overline{\theta}]
\end{aligned}
\]  

(8)
Case of nonlinearities on the outputs 1/6

Given a system described by:

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\dot{x} &= Ax(t) + \varphi(y(t), \theta, u(t)) \\
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\end{align*}
\] (7)

▷ Assumption 1: There exists a gain \( L \) such that \((A - LC)\) is Hurwitz and Metzler, i.e. \((A - LC) \in \mathcal{H} \& \mathcal{M}\).

▷ Assumption 2: There exist two bracketing functions \( \varphi, \overline{\varphi} \) and a positive real vector \( M < +\infty \) such that:

\[
\begin{align*}
\varphi(y(t), \theta_m, \theta_M, u(t)) &\leq \varphi(y(t), \theta, u(t)) \leq \overline{\varphi}(y(t), \theta_m, \theta_M, u(t)) \\
\|\overline{\varphi}(y(t), \theta_m, \theta_M, u(t)) - \varphi(y(t), \theta_m, \theta_M, u(t))\| &\leq M
\end{align*}
\]
∀(x, u, \( \theta \)) ∈ \( \mathcal{D} \times \mathcal{U} \times [\theta, \overline{\theta}] \) (8)
Case of nonlinearities on the outputs 1/6

Given a system described by:

\[
\begin{aligned}
\dot{x} &= Ax(t) + \varphi(y(t), \theta, u(t)) \\
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\end{aligned}
\]  \hspace{1cm} (7)

- **Assumption 1:** There exists a gain \( L \) such that \((A - LC)\) is Hurwitz and Metzler, i.e. \((A - LC) \in \mathcal{H} \& \mathcal{M}\).

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\|
\underline{\varphi}(y(t), \theta_m, \theta_M, u(t)) - \varphi(y(t), \theta_m, \theta_M, u(t))
\| &\leq M \\
\forall (x, u, \theta) \in \mathcal{D} \times \mathcal{U} \times [\theta, \overline{\theta}]
\end{aligned}
\]  \hspace{1cm} (8)
Remark

- *The bounding functions* $\varphi$ and $\overline{\varphi}$ *do not contain any uncertainty.*

- Monotony properties with respect to $\theta$ are usually used to build $\varphi$, $\overline{\varphi}$. 
Case of nonlinearities on the outputs 2/6

Remark

- The bounding functions $\underline{\varphi}$ and $\overline{\varphi}$ do not contain any uncertainty.

- Monotony properties with respect to $\theta$ are usually used to build $\underline{\varphi}$, $\overline{\varphi}$. 
Case of nonlinearities on the outputs 3/6

Interval observer structure

\[
\begin{align*}
\dot{\underline{x}} &= A\underline{x} + \underline{\varphi}(y, \theta_m, \theta_M, u) + L (y - C\underline{x}) \\
\dot{\overline{x}} &= A\overline{x} + \overline{\varphi}(y, \theta_m, \theta_M, u) + L (y - C\overline{x}) \\
x(t_0) &\leq x(t_0) \leq \overline{x}(t_0)
\end{align*}
\]

\[\Rightarrow \quad \underline{x}(t) \leq x(t) \leq \overline{x}(t)\]
Case of nonlinearities on the outputs 3/6

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\begin{align*}
\dot{\underline{x}} &= A\underline{x} + \underline{\varphi}(y, \theta_m, \theta_M, u) + L (y - C\underline{x}) \\
\dot{\overline{x}} &= A\overline{x} + \overline{\varphi}(y, \theta_m, \theta_M, u) + L (y - C\overline{x}) \\
\underline{x}(t_0) &\leq x(t_0) \leq \overline{x}(t_0)
\end{align*}
\]

\[\Rightarrow \quad \underline{x}(t) \leq x(t) \leq \overline{x}(t)?\]
Introduction

Case of nonlinearities on the outputs 4/6

Proof - inclusion property
The lower error $e(t) = x(t) - \underline{x}(t)$ dynamics is described by:

$$\dot{e}(t) = (A - LC) e(t) + b(t)$$

(10)

with $b(t) = \varphi(y(t), \theta, u(t)) - \underline{\varphi}(y(t), \theta_m, \theta_M, u(t)) \geq 0$.

- With respect to Assumption 1, the gain $L$ is chosen such that $(A - LC) \in \mathcal{M}$.
- By construction, $e(t_0) = x(t_0) - \underline{x}(t_0) \geq 0$.
- $\Rightarrow e(t) \geq 0, \forall t \geq t_0$.
- $\Rightarrow x(t) \geq \underline{x}(t), \forall t \geq t_0$.
- Similarly, we can easily prove that the upper error satisfies $\underline{x}(t) \leq x(t) \leq \bar{x}(t), \forall t \geq t_0$.

$$\Rightarrow \underline{x}(t) \leq x(t) \leq \bar{x}(t)$$
Case of nonlinearities on the outputs 4/6

Proof - inclusion property
The lower error $\epsilon(t) = x(t) - \bar{x}(t)$ dynamics is described by:

$$\dot{\epsilon}(t) = (A - LC) \epsilon(t) + b(t)$$ (10)

with $b(t) = \varphi(y(t), \theta, u(t)) - \varphi(y(t), \theta_m, \theta_M, u(t)) \geq 0$.

- With respect to Assumption 1, the gain $L$ is chosen such that $(A - LC) \in \mathcal{M}$.
- By construction, $\epsilon(t_0) = x(t_0) - \bar{x}(t_0) \geq 0$.
- $\Rightarrow \epsilon(t) \geq 0, \forall t \geq t_0$.
- $\Rightarrow x(t) \geq \underline{x}(t), \forall t \geq t_0$.
- Similarly, we can easily prove that the upper error satisfies $x(t) \leq \bar{x}(t), \forall t \geq t_0$.

$\Rightarrow \underline{x}(t) \leq x(t) \leq \bar{x}(t)$
Case of nonlinearities on the outputs 4/6

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(10)

with $b(t) = \varphi(y(t), \theta, u(t)) - \underline{\varphi}(y(t), \theta_m, \theta_M, u(t)) \geq 0$.

- With respect to Assumption 1, the gain $L$ is chosen such that $(A - LC) \in M$.
- By construction, $\varepsilon(t_0) = x(t_0) - \underline{x}(t_0) \geq 0$.
- $\Rightarrow \varepsilon(t) \geq 0, \forall t \geq t_0$.
- $\Rightarrow x(t) \geq \underline{x}(t), \forall t \geq t_0$.
- Similarly, we can easily prove that the upper error satisfies $x(t) \leq \overline{x}(t), \forall t \geq t_0$.

$\Rightarrow \underline{x}(t) \leq x(t) \leq \overline{x}(t)$
Proof - stability

By Assumption 1, the matrix \((A - LC)\) is Hurwitz stable.

In addition, the boundedness of \(\varphi\) implies that \(b(t)\) is also bounded \(\forall y \geq t_0\).

\[ \Rightarrow \text{the error } e \text{ is stable.} \]

\[ \ldots \text{and similarly for } \bar{e} \]
Case of nonlinearities on the outputs 6/6

Convergence

**Theorem**

Given a system described by (7), such as:

- there exists a gain $L$ with $(A - LC)$ is Metzler;
- $(A - LC)$ is invertible and its inverse is stable;
- $w \left( [\underline{\varphi}(y(t), \theta_m, \theta_M, u), \overline{\varphi}(y(t), \theta_m, \theta_M, u)] \right) \leq B$.

$\Rightarrow$ The total error $\bar{e}(t) + e(t)$ converges asymptotically (componentwise) to: $e_{\text{max}} = -(A - LC)^{-1}B$. 

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Given a system described by:

\[
\begin{cases}
    \dot{x} = Ax(t) + \varphi(x(t), y(t), \theta, u(t)) \\
    y = Cx(t)
\end{cases}
\]  

(11)

- **Assumption 1:** There exists a gain $L$ such that $(A - LC) \in \mathcal{M} \& \mathcal{H}$.

- **Assumption 2:** There exist two bounding vector fields $\varphi, \overline{\varphi}$ and positive $M < +\infty$ such as:

\[
\begin{cases}
    \varphi(x_m, x_M, y, \theta_m, \theta_M, u) \leq \varphi(x, y, \theta, u) \leq \overline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) \\
    \|\overline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) - \varphi(x_m, x_M, y, \theta_m, \theta_M, u)\| \leq M \\
    \forall(x, u, \theta) \in \mathcal{D} \times \mathcal{U} \times [\theta, \overline{\theta}]
\end{cases}
\]

(12)
Partial linear systems 1/4

Given a system described by:

\[
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\end{cases}
\]  

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- **Assumption 1:** There exists a gain $L$ such that $(A - LC) \in \mathcal{M}_\mathcal{H}$.

- **Assumption 2:** There exist two bounding vector fields $\underline{\varphi}, \overline{\varphi}$ and positive $M < +\infty$ such as:

\[
\begin{cases}
\varphi(x_m, x_M, y, \theta_m, \theta_M, u) \leq \varphi(x, y, \theta, u) \leq \overline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) \\
\|\overline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) - \varphi(x_m, x_M, y, \theta_m, \theta_M, u)\| \leq M \\
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\begin{align*}
\underline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) &\leq \varphi(x, y, \theta, u) \leq \overline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) \\
\|\overline{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) - \varphi(x, y, \theta, u)\| &\leq M \\
\forall (x, u, \theta) &\in \mathcal{D} \times \mathcal{U} \times [\underline{\theta}, \overline{\theta}]
\end{align*}
\]  

(12)
Partial linear systems 2/4

How to build the bounding functions $\varphi$, $\overline{\varphi}$

✓ If $\varphi$ is monotone w.r.t $x$ and $\theta$:
  - $x_m$, $x_M$ are chosen from $\underline{x}$, $\overline{x}$ ⇒ Verify $\frac{\partial \varphi_i}{\partial x_j}$;
  - $\theta_m$, $\theta_M$ are chosen from $\underline{\theta}$, $\overline{\theta}$ ⇒ Verify $\frac{\partial \varphi_i}{\partial \theta_j}$.

✓ If $\varphi$ is not monotone w.r.t. $x$ and $\theta$:
  - $x_m$, $x_M$ correspond to some values within the interval $]x, \overline{x}[$.
  - $\theta_m$, $\theta_M$ belong into $]\underline{\theta}, \overline{\theta}[$.

⇒ The computation of $\varphi$ and $\overline{\varphi}$ would be performed online with additional computational burden.
Partial linear systems 2/4

How to build the bounding functions $\varphi$, $\bar{\varphi}$

✓ If $\varphi$ is monotone w.r.t $x$ and $\theta$:

- $x_m, x_M$ are chosen from $\underline{x}, \overline{x}$ ⇒ Verify $\frac{\partial \varphi_i}{\partial x_j}$;
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✓ If $\varphi$ is not monotone w.r.t. $x$ and $\theta$:

- $x_m, x_M$ correspond to some values within the interval $]x, \overline{x}[$.
- $\theta_m, \theta_M$ belong into $]\underline{\theta}, \overline{\theta}[$.

⇒ The computation of $\varphi$ and $\bar{\varphi}$ would be performed online with additional computational burden.
Partial linear systems 3/4

Observer structure

\[
\begin{align*}
\dot{x} &= Ax + \varphi(x_m, x_M, y, \theta_m, \theta_M, u) + L(y - Cx) \\
\dot{\bar{x}} &= A\bar{x} + \varphi(x_m, x_M, y, \theta_m, \theta_M, u) + L(y - C\bar{x}) \\
x(t_0) \leq x(t) &\leq \bar{x}(t_0)
\end{align*}
\]

\[
\Rightarrow \quad x(t) \leq \bar{x}(t) \leq \bar{x}(t)
\]
Observer structure

\[
\begin{align*}
\dot{x} &= A\bar{x} + \varphi(x_m, x_M, y, \theta_m, \theta_M, u) + L(y - C\bar{x}) \\
\dot{\bar{x}} &= A\bar{x} + \bar{\varphi}(x_m, x_M, y, \theta_m, \theta_M, u) + L(y - C\bar{x}).
\end{align*}
\]

\( \bar{x}(t_0) \leq x(t_0) \leq \bar{x}(t_0) \)

\[ \Rightarrow \quad \underline{x}(t) \leq x(t) \leq \bar{x}(t) \]
Remark

- Usually, it is hard to ensure the Metzler property of $(A - LC)$, How to do?
1. Introduction

2. Prediction/Correction approach

3. Interval observers

4. Relaxation of cooperativity - changes of coordinates
   - Time-varying systems
   - Time-varying changes of coordinates of LTV systems
   - Linear Parameter-Varying systems
Given a system described by:

\[ \dot{x} = Ax + Bu, \quad y = Cx, \]

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \]

This system is observable, whereas the matrix

\[ A - LC = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix} \]

cannot be Hurwitz and Metzler!
Changes of coordinates

Given a system described by:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]  
(14)

with the following (point) observer structure:

\[
\begin{align*}
\dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly \\
\hat{y} &= C\hat{x}
\end{align*}
\]  
(15)

Usually \( \not\exists L \mid (A - LC) \in \mathcal{H} \& \mathcal{M} \)

\( \Rightarrow \) but there always exist \( L \) and \( P \in \mathbb{R}^{n \times n} \mid R = P(A - LC)P^{-1} \in \mathcal{H}\&\mathcal{M} \)

\( \Rightarrow \) an interval observer can be designed in the new coordinates \( z = Px \).
Changes of coordinates

Given a system described by:

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\( \Rightarrow \) an interval observer can be designed in the new coordinates \( z = Px \).
Changes of coordinates

Given a system described by:

\[
\begin{cases}
\dot{x} = Ax + Bu \\
y = Cx
\end{cases}
\]  

(14)

with the following (point) observer structure:

\[
\begin{cases}
\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \\
\hat{y} = C\hat{x}
\end{cases}
\]  

(15)

Usually \( \not\exists L \mid (A - LC) \in \mathcal{H} \, \& \, \mathcal{M} \)

\[ \Rightarrow \text{but there always exist } L \text{ and } P \in \mathbb{R}^{n \times n} \mid R = P(A - LC)P^{-1} \in \mathcal{H} & \mathcal{M} \]

\[ \Rightarrow \text{an interval observer can be designed in the new coordinates } z = Px. \]
Time-invariant changes of coordinates 1/3

Given the system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]  \hspace{1cm} (16)

The idea is to compute a nonsingular matrix $P$ such that, in the coordinates $z = Px$, the system:

\[
\begin{align*}
\dot{z} &= PAP^{-1}z + PBu \\
y &= CP^{-1}z
\end{align*}
\]  \hspace{1cm} (17)

has a point (conventional) observer:

\[
\dot{\hat{z}} = PAP^{-1}\hat{z} + PBu + PL(y - CP^{-1}\hat{z})
= R\hat{z} + PBu + PLy,
\]  \hspace{1cm} (18)

where $R = PAP^{-1} - PLCP^{-1}$ is Hurwitz stable & Metzler.
Time-invariant changes of coordinates 2/3

\[ R = PAP^{-1} - PLCP^{-1} \]

\( P \) is nonsingular \( \Rightarrow \) **Sylvester equation is formulated:**

\[ PA - RP = QC, \quad Q = PL. \] (19)

**Lemma**

*Given \((A - LC)\) and \(R \in \mathcal{M}\) with the same eigenvalues. If \(\exists\) two vectors \(e_1\) and \(e_2\) such that \((A - LC, e_1)\) and \((R, e_2)\) are observable, then*\(^a\)

\[ P = O_2^{-1}O_1 \quad \text{and} \quad Q = PL, \]

satisfies (19) where

\[
O_1 = \begin{bmatrix}
  e_1 \\
  \vdots \\
  e_1(A - LC)^{n-1}
\end{bmatrix}; \quad O_2 = \begin{bmatrix}
  e_2 \\
  \vdots \\
  e_2R^{n-1}
\end{bmatrix}
\]

*\(^a\)T. Raïssi, D. Efimov, A. Zolghadri, ”Interval State Estimation for a Class of Nonlinear Systems,” IEEE Transactions on Automatic Control, 57(1), 260-265, 2012*
Time-invariant changes of coordinates 2/3

\[ R = P A P^{-1} - P L C P^{-1} \]

\( P \) is nonsingular \( \Rightarrow \) Sylvester equation is formulated:

\[ PA - RP = QC, \quad Q = PL. \] (19)

**Lemma**

Given \( (A - LC) \) and \( R \in \mathcal{M} \) with the same eigenvalues. If \( \exists \) two vectors \( e_1 \) and \( e_2 \) such that \( (A - LC, e_1) \) and \( (R, e_2) \) are observable, then\(^a\)

\[ P = O_2^{-1} O_1 \text{ and } Q = PL, \]

satisfies (19) where \( O_1 = \begin{bmatrix}  e_1 \\ \vdots \\ e_1 (A - LC)^{n-1} \end{bmatrix} \) \( ; \) \( O_2 = \begin{bmatrix}  e_2 \\ \vdots \\ e_2 R^{n-1} \end{bmatrix} \)

\(^a\)T. Raïssi, D. Efimov, A. Zolghadri, "Interval State Estimation for a Class of Nonlinear Systems," IEEE Transactions on Automatic Control, 57(1), 260-265, 2012
Sylvester formulation - practical procedure

The following simple procedure\textsuperscript{2} can be used to design a matrix $R$ with the same eigenvalues as $(A - LC)$:

- Assume that $(A, C)$ is detectable,
- Compute a gain $L$ such that $(A - LC)$ has only real eigenvalues:
  1. if $(A - LC)$ is diagonalizable $\Rightarrow R = diag(eigenvalues(A - LC))$;
  2. otherwise, we can choose a triangular matrix $R$ with the eigenvalues of $(A - LC)$ as diagonal terms.

\textsuperscript{2}T. Raïssi, D. Efimov, A. Zolghadri, Interval state estimation for a class of nonlinear systems, IEEE Transactions on Automatic Control, 57(1), 260-265, 2012.
Time-varying changes of coordinates

Second solution: Jordan canonical form\(^3,4\)

Given the system:

\[ \dot{x} = Ax \]  \hspace{1cm} (20)

It is always possible to transform (20) into a cooperative representation using the Jordan canonical form:

- time-invariant change of coordinates if the eigenvalues of \(A\) are real;
- time-varying change of coordinates if \(A\) has some complex eigenvalues.

\(^3\)F. Mazenc, O. Bernard, Interval observers for linear time-invariant systems with disturbances, Automatica, 47(1), 140-147, 2011

Extension to a class of nonlinear systems

Given a system described by:

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= g(x)
\end{align*}
\]  

(21)

If the system (21) is locally observable, then, it is possible to build a local diffeomorphism \(\Phi\) which transforms (21) into a partial linear system:

\[
\begin{align*}
\dot{\xi} &= A\xi + B(\alpha(\xi) + \beta(\xi)u) \\
y &= C\xi
\end{align*}
\]  

(22)

Usually, (22) is a canonical observable form \(\Rightarrow\) and it is not possible to compute a gain \(L\) such that \((A - LC) \in \mathcal{H} \& \mathcal{M}\).

\(\Rightarrow\) The changes of coordinates presented above can be used.
Numerical example 1/2

\[ \dot{x} = Ax + B(p_1, p_2)f(x)u(t), \quad y = Cx, \]

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
1 & -4 & \sqrt{3} \\
-1 & -\sqrt{3} & -4
\end{bmatrix}, \quad B(p_1, p_2) = \begin{bmatrix}
-2p_1 \\
0 \\
p_2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix},
\]

\[ f(x) = x_1x_2, \quad p_1 = 4.48, \quad \bar{p}_1 = 6.12, \quad p_2 = 3.2, \quad \bar{p}_2 = 3.6. \]

- The pair \((A, C)\) is not observable and there is no observer gain \(L\) such that the matrix \(A - LC\) is Metzler.
- Only one eigenvalue can be assigned with the gain \(L\).
- The matrix

\[
R = \begin{bmatrix}
-a & b & 0 \\
0 & -a & b \\
b & 0 & -a
\end{bmatrix}
\]

has the following eigenvalues \(b - a, \ -a - 0.5b \pm 0.5b\sqrt{3}i\) (we take here \(b = 2\) and \(a = 3\)).
For $L = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T$, the matrix $A - LC \in \mathcal{H}$ and its eigenvalues are $-1, -4 \pm \sqrt{3}i$.

The pairs $(A - LC, e_1)$ and $(R, e_2)$ are observable for

$$e_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

then

$$P = O_2^{-1}O_1 = \begin{bmatrix} 0.158 & 0.866 & 0.5 \\ 0.842 & -0.866 & 0.5 \\ 0.658 & 0 & -1 \end{bmatrix}.$$
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4 Relaxation of cooperativity - changes of coordinates
  - Time-varying systems
  - Time-varying changes of coordinates of LTV systems
  - Linear Parameter-Varying systems
Consider the system:

\[
\begin{aligned}
\dot{x} &= A(t, y, u)x + f(t, x, u, \varrho), \\
y &= C(t, u)x,
\end{aligned}
\]  

(23)

- \(\varrho \in \Theta \subset \mathbb{R}^q\) would be an uncertain parameter vector or unknown inputs (\(\Theta\) is known);

- \(A : \mathbb{R}^{p+m+1} \rightarrow \mathbb{R}^{n \times n}, C : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{p \times n}\) and \(f : \mathbb{R}^{n+m+q+1} \rightarrow \mathbb{R}^{n \times m}\) are available;
Assumption 3. $\|x\| \leq X$, $\|u\| \leq U$ and $\|y\| \leq Y$, where the constants $X > 0$, $U > 0$, $Y > 0$ are given.

Assumption 4. $\exists f, \bar{f}$ such that $\forall x \leq x \leq \bar{x}$, $\|u\| \leq U$, $\rho \in \Theta$, we have:

$$f(t, x, \bar{x}, u) \leq f(t, x, u, \rho) \leq \bar{f}(t, x, \bar{x}, u).$$

Assumption 5. $\exists L : \mathbb{R}^{p+m+1} \rightarrow \mathbb{R}^{n \times p}$, $P : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$, $P(\cdot) = P(\cdot)^T > 0$ such that $\forall t \geq 0$ and $\|u\| \leq U$, $\|y\| \leq Y$:

$$\dot{P}(t) + D(t, y, u)^T P(t) + P(t) D(t, y, u) + P(t)^2 + Q = 0,$$

$$D(t, y, u) = A(t, y, u) - L(t, y, u) C(t, u), \quad Q = Q^T > 0.$$
Partially Linear Time-Varying systems (p-LTV)

Assumption 3. \( ||x|| \leq X, ||u|| \leq U \) and \( ||y|| \leq Y \), where the constants \( X > 0, U > 0, Y > 0 \) are given.

Assumption 4. \( \exists f, \bar{f} \) such that \( \forall x \leq x \leq \bar{x}, ||u|| \leq U, \varrho \in \Theta \), we have:

\[
f(t, x, \bar{x}, u) \leq f(t, x, u, \varrho) \leq \bar{f}(t, x, \bar{x}, u).
\]

Assumption 5. \( \exists L : \mathbb{R}^{p+m+1} \rightarrow \mathbb{R}^{n \times p}, P : \mathbb{R}^{+} \rightarrow \mathbb{R}^{n \times n}, P(\cdot) = P(\cdot)^T > 0 \) such that \( \forall t \geq 0 \) and \( ||u|| \leq U, ||y|| \leq Y \):

\[
\dot{P}(t) + D(t, y, u)^T P(t) + P(t)D(t, y, u) + P(t)^2 + Q = 0,
\]

\[
D(t, y, u) = A(t, y, u) - L(t, y, u)C(t, u), \quad Q = Q^T > 0.
\]
Partially Linear Time-Varying systems (p-LTV)

Assumption 3. $||x|| \leq X$, $||u|| \leq U$ and $||y|| \leq Y$, where the constants $X > 0$, $U > 0$, $Y > 0$ are given.

Assumption 4. $\exists f, \bar{f}$ such that $\forall \underline{x} \leq x \leq \overline{x}$, $||u|| \leq U$, $\varrho \in \Theta$, we have:

$$f(t, \underline{x}, \overline{x}, u) \leq f(t, x, u, \varrho) \leq \bar{f}(t, \underline{x}, \overline{x}, u).$$

Assumption 5. $\exists L : \mathbb{R}^{p+m+1} \to \mathbb{R}^{n \times p}$, $P : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$, $P(\cdot) = P(\cdot)^T \succ 0$ such that $\forall t \geq 0$ and $||u|| \leq U$, $||y|| \leq Y$:

$$p_1 l_n \leq P(t) \preceq p_2 l_n, \ p_1, p_2 > 0;$$

$$\dot{P}(t) + D(t, y, u)^T P(t) + P(t) D(t, y, u) + P(t)^2 + Q = 0,$$

$$D(t, y, u) = A(t, y, u) - L(t, y, u) C(t, u), \ Q = Q^T \succ 0.$$
Partially Linear Time-Varying systems (p-LTV)

Let’s consider the following observer structure:

\[
\dot{x} = A(t, y, u)x + f(t, x, \bar{x}, u) + L(t, y, u)[y - C(t, u)x], \quad (24)
\]

\[
\dot{\bar{x}} = A(t, y, u)\bar{x} + \bar{f}(t, x, \bar{x}, u) + L(t, y, u)[y - C(t, u)\bar{x}]. \quad (25)
\]

How to ensure that \( x, \bar{x} \) are bounds of the state vector \( x \) ?
First case:  

**Theorem**

Assume that $D(t, y, u) \in \mathcal{M} \forall t \geq 0$ and that Assumptions 3-5 are satisfied. If

1. $|f(t, x, x, u)| < +\infty$, $|\overline{f}(t, x, x, u)| < +\infty$;

2. $\exists \alpha \in \mathbb{R}_+, \beta \in \mathbb{R}_+$:

$$
|f(t, x, u, \varrho) - f(t, x, x, u)|^2 + |\overline{f}(t, x, x, u) - f(t, x, u, \varrho)|^2 \\
\leq \beta|x - x|^2 + \beta|x - \overline{x}|^2 + \alpha,
$$

Then, the variables $\underline{x}(t)$ and $\overline{x}(t)$ (in (24), (25)) are bounded and verify

$$
\underline{x}(0) \leq x(0) \leq \overline{x}(0) \implies \underline{x}(t) \leq x(t) \leq \overline{x}(t)
$$

---

5. D. Efimov, T. Raïssi, S. Chebotarev, A. Zolghadri, Interval state observer for nonlinear time-varying systems, Automatica, 49(1), 200-205, 2013
First case:  

**Theorem**

Assume that $D(t, y, u) \in M \ \forall \ t \geq 0$ and that Assumptions 3-5 are satisfied. If

1. $|f(t, x, \overline{x}, u)| < +\infty$, $|\overline{f}(t, x, \overline{x}, u)| < +\infty$;
2. $\exists \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+$:

\[
|f(t, x, u, \varrho) - f(t, x, \overline{x}, u)|^2 + |\overline{f}(t, x, \overline{x}, u) - f(t, x, u, \varrho)|^2 \\
\leq \beta|x - \overline{x}|^2 + \beta|\overline{x} - x|^2 + \alpha,
\]

Then, the variables $\underline{x}(t)$ and $\overline{x}(t)$ (in (24), (25)) are bounded and verify

$\underline{x}(0) \leq x(0) \leq \overline{x}(0) \Rightarrow \underline{x}(t) \leq x(t) \leq \overline{x}(t)$

---

5D. Efimov, T. Raïssi, S. Chebotarev, A. Zolghadri, Interval state observer for nonlinear time-varying systems, Automatica, 49(1), 200-205, 2013
Partially Linear Time-Varying systems (p-LTV)

First case:\textsuperscript{5}

\textbf{Theorem}

Assume that $D(t, y, u) \in \mathcal{M} \forall t \geq 0$ and that Assumptions 3-5 are satisfied. If

1. $|f(t, \underline{x}, \overline{x}, u)| < +\infty$, $|\overline{f}(t, \underline{x}, \overline{x}, u)| < +\infty$;
2. $\exists \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+$:

\[
|f(t, x, u, \varrho) - f(t, \underline{x}, \overline{x}, u)|^2 + |\overline{f}(t, \underline{x}, \overline{x}, u) - f(t, \underline{x}, \overline{x}, u)|^2 \\
\leq \beta |x - \underline{x}|^2 + \beta |\overline{x} - x|^2 + \alpha,
\]

Then, the variables $\underline{x}(t)$ and $\overline{x}(t)$ (in (24), (25)) are bounded and verify

$\underline{x}(0) \leq x(0) \leq \overline{x}(0) \Rightarrow \underline{x}(t) \leq x(t) \leq \overline{x}(t)$

\textsuperscript{5}D. Efimov, T. Raïssi, S. Chebotarev, A. Zolghadri, Interval state observer for nonlinear time-varying systems, Automatica, 49(1), 200-205, 2013
Second case: 6

Lemma 2. Let $D \in \Xi \subset \mathbb{R}^{n \times n}$, where

$$\Xi = \{ D \in \mathbb{R}^{n \times n} : D_a - \Delta \leq D \leq D_a + \Delta \}, \quad D_a^T = D_a \in \mathbb{R}^{n \times n}, \quad \Delta \in \mathbb{R}^{n \times n}_+. $$

If $\exists \mu \in \mathbb{R}$ and a diagonal matrix $\Upsilon \in \mathbb{R}^{n \times n}$

$$R = \mu E_n - \Upsilon \in \mathcal{M} \quad \text{and} \quad \lambda(R) = \lambda(D_a).$$

Then, there exists an orthogonal matrix $S \in \mathbb{R}^{n \times n}$ such that

$$S^T DS \in \mathcal{M} \quad \forall D \in \Xi$$

provided that $\mu > n \| \Delta \|_{\max}.$

---

6D. Efimov, T. Ra"issi, S. Chebotarev, A. Zolghadri, Interval state observer for nonlinear time-varying systems, Automatica, 49(1), 200-205, 2013
System transformation & observer structure
In the coordinates $z = S^T x$, the time-varying system (23) can be transformed into:

$$\dot{z} = S^T A(t, y, u) S z + \phi(t, z, u, \varphi), \quad \phi(t, z, u, \varphi) = S^T f(t, S z, u, \varphi)$$

$$z \leq z \leq \overline{z} :$$

$$\underline{x} = S^+ z - S^- \overline{z} \leq x = S z \leq S^+ \overline{z} - S^- z = \overline{x},$$

$$\underline{\phi}(t, z, \overline{z}, u) = S^+ T f(t, \underline{x}, \overline{x}, u) - S^- T \overline{f}(t, \underline{x}, \overline{x}, u) \leq \phi(t, z, u, \varphi)$$

$$\leq S^+ T \overline{f}(t, \underline{x}, \overline{x}, u) - S^- T f(t, \underline{x}, \overline{x}, u) = \overline{\phi}(t, z, \overline{z}, u).$$

Interval observer structure:

$$\dot{\underline{z}} = S^T A(t, y, u) S \underline{z} + \underline{\phi}(t, \underline{z}, \overline{z}, u) + S^T L(t, y, u) [y - C(t, u) S \underline{z}]$$

$$\dot{\overline{z}} = S^T A(t, y, u) S \overline{z} + \overline{\phi}(t, \underline{z}, \overline{z}, u) + S^T L(t, y, u) [y - C(t, u) S \overline{z}].$$

The restrictive Metzler condition of $D(t)$ is relaxed by the means of a change of coordinates.
**System transformation & observer structure**

In the coordinates \( z = S^T x \), the time-varying system (23) can be transformed into:

\[
\dot{z} = S^T A(t, y, u)Sz + \phi(t, z, u, \varphi), \quad \phi(t, z, u, \varphi) = S^T f(t, Sz, u, \varphi)
\]

\[
\begin{align*}
  z &\leq z &\leq \bar{z}:
  x &= S^+ z - S^- \bar{z} &\leq x &= Sz &\leq S^+ \bar{z} - S^- z &= \bar{x},
  \phi(t, z, \bar{z}, u) &= S^+ f(t, x, \bar{x}, u) - S^- f(t, x, \bar{x}, u) \\
  \leq S^+ f(t, x, \bar{x}, u) - S^- f(t, x, \bar{x}, u) &= \bar{\phi}(t, z, \bar{z}, u).
\end{align*}
\]

**Interval observer structure:**

\[
\begin{align*}
  \dot{z} &= S^T A(t, y, u)Sz + \underline{\phi}(t, z, \bar{z}, u) + S^T L(t, y, u)[y - C(t, u)Sz],
  \bar{z} &= S^T A(t, y, u)S\bar{z} + \overline{\phi}(t, z, \bar{z}, u) + S^T L(t, y, u)[y - C(t, u)S\bar{z}].
\end{align*}
\]

The restrictive Metzler condition of \( D(t) \) is relaxed by the means of a change of coordinates.
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LTV systems models

\begin{align}
\dot{x}(t) &= A(t)x(t) + f(t) \\
y(t) &= C(t)x(t) + \varphi(t)
\end{align}

(28)

**Assumption 6.**

There exist bounded matrix functions $L_{\text{obs}} : \mathbb{R} \to \mathbb{R}^{n \times p}$, $M : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$, $M(\cdot) = M(\cdot)^T \succ 0$ such that for all $t \geq 0$,

\begin{align}
\dot{M}(t) + D(t)^T M(t) + M(t)D(t) &< 0, \\
D(t) = A(t) - L_{\text{obs}}(t)C(t).
\end{align}

Then, the system (1) can be rewritten as:

\begin{align}
\dot{x}(t) &= D(t)x(t) + \tilde{\varphi}(t) \\
y(t) &= C(t)x(t) + \varphi(t)
\end{align}

(29)
D-Similarities

**Lemma**

Two time-varying matrices $A_1(t) \in \mathbb{R}^{n \times n}$ and $A_2(t) \in \mathbb{R}^{n \times n}$ are said to be D-similar if there exists a transformation matrix $\Sigma(t) \in \mathbb{R}^{n \times n}$ such that $\det(\Sigma(t)) \equiv constant \neq 0$ and

$$
A_2(t) = \Sigma^{-1}(t) [A_1(t)\Sigma(t) - \dot{\Sigma}(t)].
$$

(30)

$A_1(t)$ and $A_2(t)$ are D-similar if $tr(A_1(t)) = tr(A_2(t))^a$.

$\det(.)$ and $tr(.)$ respectively denote the determinant and the trace of a square matrix.

---

A procedure proposed in (Zhu & Johnson, 1991) shows that it is possible to determine a D-Similarity, with a transformation matrix $T(t)$, between $D(t)$ and a Metzler matrix $\Gamma(t)$ described by:

$$\Gamma(t) = \begin{bmatrix}
\lambda_1(t) & 1 & \cdots & 0 \\
0 & \lambda_2(t) & \cdots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & \lambda_n(t)
\end{bmatrix},$$

(31)

where $\lambda_i(t)$ are Essential D-eigenvalues (or ED-eigenvalues, see Zhu & Johnson, 1991) of $D(t)$. 

**Time-Varying transformation of time-varying matrices**
Observer structure

Theorem

Given the system (28) and a matrix $T$ transforming $D(t)$ into $\Gamma(t)$, let Assumption 6 be satisfied and the matrix $T$ and its inverse be componentwise bounded.

Then, the system (32) is an interval observer for (29) in the coordinates $z(t) = T(t)x(t)^a$:

\[
\begin{aligned}
\dot{z}(t) &= \Gamma(t)z(t) + \phi_{obs}(t) + \Psi_{obs}(t) + T_{obs}(t)y(t) \\
\ddot{z}(t) &= \Gamma(t)\bar{z}(t) + \phi_{obs}(t) + \bar{\Psi}_{obs}(t) + T_{obs}(t)y(t)
\end{aligned}
\]  

(32)

\[
z(t) \leq T(t)x(t) \leq \bar{z}(t), \quad \forall t \geq t_0,
\]  

(33)

where $\phi_{obs}(t) = T^+(t)f(t) - T^-(t)\bar{f}(t)$, $T_{obs}(t) = T(t)L_{obs}(t)$,

\[
\begin{aligned}
\Psi_{obs}(t) &= T^-_{obs}(t)\varphi(t) - T^+_{obs}(t)\bar{\varphi}(t), \\
\bar{\phi}_{obs}(t) &= T^+(t)\bar{f}(t) - T^-(t)f(t), \\
\bar{\Psi}_{obs}(t) &= T^-_{obs}(t)\bar{\varphi}(t) - T^+_{obs}(t)\varphi(t).
\end{aligned}
\]

\(^a\)R.H. Thabet, T. Raiïssi, C. Combastel, D. Efimov, A. Zolghadri, An effective method to interval observer design for time-varying systems, Automatica, to appear.
1 Introduction

2 Prediction/Correction approach

3 Interval observers
   - LTI systems
   - Partial linear systems

4 Relaxation of cooperativity - changes of coordinates
   - Time-varying systems
   - Time-varying changes of coordinates of LTV systems
   - Linear Parameter-Varying systems
Interval observer structure - 1/2

Given a LPV system described by:

\[
\dot{x} = [A_0 + \Delta A(\rho(t))]x + b(t), \quad y = Cx
\]

Denote by: \( x^+ = \max(x, 0), \quad x^- = x^+ - x, \quad A^+ = \max(A, 0), \quad A^- = A^+ - A, \Rightarrow \) only nonnegative vectors and matrices are used.

Observer structure

\[
\dot{x} = [A_0 - LC]x + \left[ \Delta A^+ x^+ - \Delta A^- x^- - \Delta A^+ x^- + \Delta A^- x^+ \right] + Ly + b(t)
\]

\[
\dot{x} = [A_0 - LC]\bar{x} + \left[ \Delta A^+ \bar{x}^+ - \Delta A^+ \bar{x}^- - \Delta A^- \bar{x}^+ + \Delta A^- \bar{x}^- \right] + Ly + \bar{b}(t)
\]
Theorem

Assume that the state $x$ is bounded and that $(A_0 - LC) \in \mathcal{M}$. Then, the observer structure proposed in the previous slide is an interval observer for the LPV system if $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. In addition, if there exist a matrix $P \in \mathbb{R}^{n \times n}$, $P = P^T > 0$ and $\gamma > 0$ such that the following Riccati equation is satisfied:

$$G^T P + PG + 2\gamma^{-2}P^2 + 4\gamma^2 \mu^2 I_{2n} + Z^T Z < 0$$

then, $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$. Moreover, the transfert $\begin{bmatrix} b \\ \tilde{b} \end{bmatrix} \rightarrow Z \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$ has a gain less to $\gamma$ in the $\mathcal{L}_\infty$ norm framework.

with $\mu = \| \Delta A - \Delta A \|_2$, $Z \in \mathbb{R}^{s \times 2n}$, $0 < s \leq 2n$ et

$$G = \begin{bmatrix} A_0 - LC + \Delta A^+ & -\Delta A^- \\ -\Delta A^- & A_0 - LC + \Delta A^+ \end{bmatrix}.$$
Theorem

Assume that the state $x$ is bounded and that $(A_0 - LC) \in \mathcal{M}$. Then, the observer structure proposed in the previous slide is an interval observer for the LPV system if $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. In addition, if there exist a matrix $P \in \mathbb{R}^{n \times n}$, $P = P^T > 0$ and $\gamma > 0$ such that the following Riccati equation is satisfied:

$$
G^T P + PG + 2\gamma^{-2}P^2 + 4\gamma^2\mu^2 l_{2n} + Z^T Z < 0
$$

then, $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$. Moreover, the transfer $\begin{bmatrix} b \\ b \end{bmatrix} \rightarrow Z \begin{bmatrix} x \\ \overline{x} \end{bmatrix}$ has a gain less to $\gamma$ in the $\mathcal{L}_\infty$ norm framework.

with $\mu = ||\overline{\Delta A} - \Delta A||_2$, $Z \in \mathbb{R}^{s \times 2n}$, $0 < s \leq 2n$ et

$$
G = \begin{bmatrix}
A_0 - LC + \Delta A^+ & -\Delta A^- \\
-\Delta A^- & A_0 - LC + \Delta A^+
\end{bmatrix}.
$$
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Assume that the state $x$ is bounded and that $(A_0 - LC) \in \mathcal{M}$. Then, the observer structure proposed in the previous slide is an interval observer for the LPV system if $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. In addition, if there exist a matrix $P \in \mathbb{R}^{n \times n}$, $P = P^T > 0$ and $\gamma > 0$ such that the following Riccati equation is satisfied:

$$G^T P + PG + 2\gamma^{-2}P^2 + 4\gamma^2 \mu^2 I_{2n} + Z^T Z \prec 0$$

then, $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$. Moreover, the transfer $\begin{bmatrix} b \\ b \end{bmatrix} \rightarrow Z \begin{bmatrix} x \\ \overline{x} \end{bmatrix}$ has a gain less to $\gamma$ in the $\mathcal{L}_\infty$ norm framework.

with $\mu = \|\overline{\Delta A} - \underline{\Delta A}\|_2$, $Z \in \mathbb{R}^{s \times 2n}$, $0 < s \leq 2n$ et

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$$G^T P + PG + 2\gamma^{-2}P^2 + 4\gamma^2\mu^2I_{2n} + Z^T Z < 0$$

then, $\underline{x}, \bar{x} \in \mathbb{L}_\infty^n$. Moreover, the transfer $\begin{bmatrix} b \\ b \end{bmatrix} \rightarrow Z \begin{bmatrix} x \\ \bar{x} \end{bmatrix}$ has a gain less to $\gamma$ in the $\mathbb{L}_\infty$ norm framework.

with $\mu = \|\Delta A_{-} - \Delta A\|_2$, $Z \in \mathbb{R}^{s \times 2n}$, $0 < s \leq 2n$ and

$$G = \begin{bmatrix} A_0 - LC + \Delta A^+ & -\Delta A^- \\ -\Delta A^- & A_0 - LC + \Delta A^+ \end{bmatrix}.$$
Conclusion

✓ Prediction / correction:
  - No observer gain design;
  - The stability cannot be ensured \textit{a priori} when dealing with large uncertainties;
  - The convergence rate cannot be tuned.

✓ Interval observers:
  - The observer gain should be designed: LMI, BMI, or NL equations;
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