Primitives for Smoothing Mobile Robot Trajectories *

Sara Fleury, Philippe Souères, Jean-Paul Laumond, Raja Chatila

LAAS-CNRS, 7 Avenue du Colonel Roche, 31077 Toulouse, France
e-mail: {sara, soueres, jpl, raja}@laas.fr

Abstract: Clothoids are very useful for smoothing the motion of a mobile robot moving along a trajectory. This paper addresses the problem of smoothing the mobile robot motions when cusps, i.e., changes of motion direction along the trajectory, are imposed. We pinpoint some special curves (that we call "anticlothoids") and we discuss how they can be used together with clothoids in order to smooth a predefined trajectory.

1 Introduction

Clothoids (or Cornu spirals) are known as very useful curves to smooth trajectories. Their equation is $\kappa = k_s s + k_0$ where $\kappa$ is the curvature, $s$ the arc length, $k_0$ the initial curvature and $k_s$ a constant characterizing the shape of the clothoid. Clothoids allow to link curves of infinite radius of curvature (i.e., lines) and curves of finite radius of curvature, with a continuous change of the curvature. Clothoids have practical applications in railway and highway design. They have been introduced in Robotics ([5, 2, 9]) for smoothing the motions of a mobile robot moving "forward" on a broken line (i.e., without a change of orientation along the trajectory).

This paper pinpoints the involutes of circles\(^1\). The natural equation of an involute of a circle is $\rho^2 = 2k_s s$, where $\rho$ is the radius of curvature and $k_s$ the radius of the circle. Another general form of this equation is $\rho = k_s (\theta - \theta_0) + \rho_0$ where $\theta$ is the direction of the tangent, $\theta_0$ the initial direction of the curve\(^2\) and $\rho_0$ the initial radius of curvature. $k_s$ is the characteristic parameter of the involute. Such curves allow to link "curves" of infinite curvature (i.e., curves reduced to a point) and curves of finite curvature, with a continuous change of the curvature. This property of the circle involutes leads us to call them anticlothoids in the context of this paper. Like clothoids, anticlothoids are the time-optimal trajectories of a two driving wheels mobile robot [6]. Both types of curves are dual from the point of view of control. They are produced by applying respectively a same constant acceleration on both wheels (anticlothoids) or constant and opposite accelerations (clothoids). The purpose of this paper is to show how to use anticlothoids in order to smooth the motions of a mobile robot when cusps are imposed (i.e., when the robot has to change the direction of its motion).

We shall first overview some known results on trajectory smoothing, mainly using clothoids ($\S$2). Then we introduce both types of curves from a control theory viewpoint, and we show how a mobile robot can execute motions supported by them ($\S$3).

Geometric properties are then proven ($\S$4). Two connected oriented straight line segments being given as a reference trajectory, we show how to plan a motion such that:

1. the velocities of the driving wheels are continuous and never simultaneously null (i.e the motion is smooth),
2. the trajectory never lies farther than some fixed threshold from the segments (this condition is required for collision avoidance),
3. the produced trajectory respects the imposed directions of motion.

The trajectories will consist of sequences of clothoids and anticlothoids.

In a last section ($\S$5), we shall conclude by sketching a method for smoothing a mobile robot's trajectory consisting of a polygonal line, while accounting for non-collision and direction changes.

2 Related work

Clothoids are extensively used for the design of highways [4] in order to smoothly join straight lines with circular portions. They are also used as splines in Computer Aided Design (CAD) [12].

They were more recently introduced in Robotics. Indeed, most trajectory planners produce trajectories consisting of straight lines and turns that force the robot to stop, because of the discontinuity in the angular speed when it has to change direction. In order to eliminate these stops, smoothing of the reference trajectory to produce a swift motion has been a long-time objective, for which the use of clothoids is very popular.

The smoothing problem was also addressed for producing directly a trajectory that joins a sequence of robot configurations $(x, y, \theta)$ smoothly, i.e., without stopping.
This problem has been addressed by Iijima, Kanayama and Yuta [5] where clothoid curves were exhibited as transition trajectories. The use of clothoids was also studied and implemented on the mobile robot Hilare [2, 10] for smoothing a path consisting of straight lines and turns.

In [9], Kanayama and Miyake join initial and final configurations by a broken line that is used as an entry to a smoothing algorithm which produces sequences of clothoid arcs and straight lines.

Shin and Singh [15] address a similar problem with the additional constraint of a lower bounded gyration radius. They produce a smooth path composed of clothoid arcs only.

In [8], Kanayama and Hartman propose the use of cubic spirals for obtaining smooth trajectories linking a sequence of configurations two by two. They propose a criterion for smoothness (a quadratic function of the curvature or of its derivative). The use of cubic spirals produces trajectories of a larger curvature radius than in the case of clothoids, thus producing a "smoother" motion.

Delingette et al. [3] generalize the smoothing problem to take into account given end conditions as well as a limited gyration radius by adding control points to the trajectory. This is much similar to the approaches in CAD. They introduced "intrinsic splines", curves whose curvature is a polynomial function of the arc length. These curves are generalizations of the clothoids and cubic spirals.

The involutes of circles were never exploited for smoothing purposes, to our knowledge. They actually correspond to the smoothing of cusps (seldom encountered in highways . . .). In robotics however, the generation of collision-free trajectories sometimes imposes such constraints. As we shall see, the involute of a circle then becomes a very useful curve.

3 Clothoids and anticlothoids

3.1 Time-optimal trajectories for a two driving wheel mobile robot

Let us consider mobile robots whose locomotion system consists of two parallel independant driving wheels (e.g., the three mobile robots of the Hilare family developed at Laas [1]). The length of the axis supporting the wheels is noted d. The reference point of the robot is the midpoint of the axis. Its coordinates are denoted by (x, y). The direction of the vehicle is noted θ. Therefore, a configuration of the robot is a triple (x, y, θ) ∈ (R² × S¹). We denote by v₀ and v₁ the linear velocities of the right and left wheels respectively (see Figure 1). The robot can then be modeled by the following control system:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{θ} \\
\dot{v}_r \\
\dot{v}_l
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}(v_r + v_l) \cos θ \\
\frac{1}{2}(v_r + v_l) \sin θ \\
\frac{1}{d}(v_r - v_l) \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
u_r \\
u_l
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} \begin{pmatrix}
u_r \\
u_l
\end{pmatrix}
\]

Moreover, the accelerations u₀ and u₁ of the wheels (i.e. the inputs of the system) are assumed to be bounded : |u₀| ≤ a and |u₁| ≤ a.

By applying the Minimum Principle (from optimal control theory [13]), one can prove that the time-optimal controls of the above system verify |u₀| = |u₁| = a [6]. Thus the time-optimal trajectories are supported by two types of curves corresponding respectively to the cases u₀ = u₁ and u₀ = -u₁.

Remarks:

1. This result does not characterize the shape of the time-optimal trajectories linking two given configurations. It just gives necessary conditions for a trajectory to be time optimal (i.e., it shows how the optimal trajectory looks like locally). At this moment, providing sufficient conditions of optimality is still an open problem (see [14] for a numerical approach to the problem).

2. We can relate v₀ and v₁ to the linear and angular accelerations v and ω by the following equation:

\[
\begin{pmatrix}
v \\
ω
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/2 \\
1/d & -1/d
\end{pmatrix} \begin{pmatrix}
u_r \\
u_l
\end{pmatrix}
\]

Thus, the time-optimal trajectories presented here are not only valid for the two-driving wheels mobile robots, but also for any system with linear and angular acceleration controls, without constraints on the curvature.

Now we show that the two types of curves have dual geometrical properties, as announced in the introduction.

Notations : A motion of the robot consists of two mappings u₀ and u₁ from a time interval [0, 1] onto \{-a, a\}. The corresponding trajectory of the robot (i.e., the locus of the points (x, y) in the plane) is noted γ. γ is a mapping from [0, 1] onto the plane R². The arc length from γ(0) to γ(t) is noted s(t). The curvature at a point γ(t) is noted κ(t), while the radius of curvature is ρ(t) = 1/κ(t). The linear and angular velocities are respectively noted v(t) and ω(t).

The direction of the tangent to γ at some point γ(t) (when it is defined) is the direction θ of the vehicle at this point, i.e., ω(t) = θ(t). Finally recall that ρ(t) = v(t)/ω(t).

All the initial values v(0), ω(0), θ(0) . . . are noted v₀, ω₀, θ₀ . . . The final values are noted v₁, ω₁, θ₁ . . .
3.2 Case \( u_r = -u_l \): the clothoids

Let us consider two constant controls verifying

\[
ur(t) = -ul(t).
\]

In this case \( v(t) = \frac{1}{2}(v_r(t) + v_l(t)) = 0 \). \( v(t) \) is constant and equal to \( v_0 \). Then \( s(t) = v_0 t \).

Now, \( v_r(t) = \pm \alpha \) while \( v_l(t) = \mp \alpha \). By integration, \( v_r(t) = \pm \alpha t + v_{r0}, v_l(t) = \mp \alpha t + v_{l0} \) and \( \omega(t) = \theta(t) = \pm \frac{\alpha}{v_0} t + \omega_0 \).

Therefore,

\[
\kappa(t) = \frac{\pm \alpha}{v_0} t + \omega_0 = \frac{\pm \alpha}{v_0} \theta(t) + \omega_0
\]

The curve is a clothoid whose characteristic constant \( k_c \) is \( \frac{\pm \alpha}{v_0} \).

When \( \omega_0 = \theta_0 = 0 \), the coordinates of a point \( \gamma(t) \) are given by the Fresnel integrals (see [7] for instance):

\[
x(t) = \text{sgn}(v_0) \int_0^t \sqrt{\frac{2 \alpha u}{k_c}} \cos \frac{\pi}{2} u^2 du
\]

\[
y(t) = \text{sgn}(v_0) \int_0^t \sqrt{\frac{2 \alpha u}{k_c}} \sin \frac{\pi}{2} u^2 du
\]

The tangent direction at this point is:

\[
\theta(t) = \text{sgn}(v_r) \frac{\alpha}{d} t^2
\]

Figure 2 shows a clothoid curve for \( u_r > 0 \). For an initial configuration \((0, 0, 0)\) the upper part is described for \( v_0 > 0 \), and the lower part for \( v_0 < 0 \). A curve symmetrical with respect to the Oz axis is obtained for \( u_r < 0 \). Computation of the coordinates when \( \omega_0 \neq 0 \) is easily done from the above system.

Remarks:

1. In the computations above, we assume \( v_0 \neq 0 \). When \( v_0 = 0 \), direct computations show that \( \gamma \) corresponds to a rotation\(^3\). Rotations can then be viewed as degenerated clothoids with infinite characteristic constants (i.e., points).

2. The linear velocity \( v \) along a clothoid is constant.

3. Equation (1) shows that clothoids can link smoothly curves of zero curvature (lines) and curves with non zero curvature.

3.3 Case \( u_r = u_l \): the anticlothoids

Now, let us consider two constant controls verifying

\[
ur(t) = ul(t).
\]

In this case, we have \( \omega(t) = \frac{1}{2}(v_r(t) - v_l(t)) = 0 \) and \( \omega(t) \) is constant and equal to \( \omega_0 \). Thus, \( \theta(t) = \omega_0 t + \theta_0 \).

The acceleration \( \dot{v}(t) \) equals \( \text{sgn}(u_r) a \). Thus, \( v(t) = \text{sgn}(u_r) a t + v_0 \).

\[
\rho(t) = \frac{v(t)}{\omega(t)} = \frac{\text{sgn}(u_r) a t + v_0}{\omega_0} = \frac{\text{sgn}(u_r) a}{\omega_0} (\theta(t) - \theta_0) + \frac{v_0}{\omega_0}
\]

The curve is an anticlothoid whose characteristic constant \( k_c \) is \( \frac{a}{\omega_0^2} \).

When \( v_0 = \theta_0 = 0 \), the coordinates of a point \( \gamma(t) \) are given by the following parametric system:

\[
x(t) = \text{sgn}(u_r) k_a (\cos(\omega_0 t) + \omega_0 t \sin(\omega_0 t) - 1)
\]

\[
y(t) = \text{sgn}(u_r) k_a (\sin(\omega_0 t) - \omega_0 t \cos(\omega_0 t))
\]

The tangent direction at this point is:

\[
\theta(t) = \omega_0 t
\]

Figure 3 shows an anticlothoid curve for \( u_r > 0 \). For an initial configuration \((0, 0, 0)\), a positive \( \omega_0 \) causes a counterclockwise motion, whereas a negative \( \omega_0 \) causes a clockwise motion. A curve symmetrical with respect to the origin is obtained for \( u_r < 0 \). The computation of the coordinates for the general case \((v_0 \neq 0, \theta_0 \neq 0)\) is given in the appendix.

![Figure 2: A clothoid for \( u_r > 0 \). Direction of motion along the curve is determined by the sign of \( v_0 \).](image)

![Figure 3: An anticlothoid for \( u_r > 0 \).](image)

\(^3\)In this paper, a rotation always designates a turn around the robot's own center (axis).
Remarks:
1. When $\omega_0 = 0$, $\gamma$ is a line. Lines appear as degenerated anticlothoids with infinite characteristic constants.
2. The angular velocity $\omega$ along an anticlothoid is constant.
3. Equation (5) shows that anticlothoids can link smoothly curves with zero radius of curvature to curves with a non zero curvature radius.
4. While anticlothoids and clothoids are dual curves from the control theory point of view, the first ones are analytic curves and the second ones are not.

3.4 How to link clothoids and anticlothoids?
Any motion starts with $v_0 = \omega_0 = 0$ and ends with $v_1 = \omega_1 = 0$. This means that any motion provided by optimal controls starts and ends either with a rotation or a translation.

Moreover, since the angular velocity is zero along a line, the trajectory portion after or before a line, if any, is necessarily a clothoid. In the same way, the portion after or before a rotation is necessarily an anticlothoid.

Figure 4 shows all the possible combinations linking clothoids and anticlothoids. The type of trajectories are the nodes of the graph, while the conditions of switching are the arcs.

![Figure 4: Possible combinations of clothoids and anticlothoids.](image)

Now, let us consider a trajectory $\gamma$ consisting of an arc of clothoid followed by an arc of anticlothoid, such that $\omega_0 = 0$ and $v_1 = 0$. Such a trajectory links a translation and a rotation (we fix $\theta_0 = 0$). Let $\tau \in [0,1]$ such that $\gamma(\tau)$ is the meeting point of the two arcs. We note $\theta(\tau), \omega(\tau), v(\tau)$ respectively $\theta_r, \omega_r$, and $v_r$.

With $u_r > 0$, we have from equation (4), $\theta_r = \frac{3}{4} \tau^2$.

From equation (8), $\theta_1 = (1 - \tau) \omega_r + \theta_r$. Because the linear velocity is constant along a clothoid, $v_r = v_0$. Now, $v_1 = 0$, thus $v_0 = (1 - \tau) a$. Because the angular velocity is constant along an anticlothoid and $\omega_0 = 0$, we have $\omega_1 = \frac{a}{\omega_r}$.

The characteristic constants of the clothoid and the anticlothoid are respectively $k_c = \frac{4a}{\omega_r}$ and $k_a = \frac{4a}{\omega_r}$.

Thus:

$$\theta_\tau = \frac{d\omega^2}{4a} = \frac{d}{4k_a}$$

$$\theta_1 = \theta_\tau + \frac{\omega_1 v_0}{a} = \frac{d}{4k_a} + \sqrt{\frac{2}{dk_a}}$$

Equation (10) gives the angular variation along $\gamma$ as a function of the characteristic parameters $k_a$ and $k_c$. It is used for proving the correctness of the smoothing primitives below.

4 Four Primitives for Smoothing Paths and their Geometric Properties
Consider a collision-free trajectory consisting of a sequence of straight-line segments. All the segments are directed. The direction of the segments gives the direction of the movement (forward or backward) on these segments. The endpoints of the segments are also directed according to the direction of the rotations at these points. Such a trajectory can be a priori given to the robot by a user or provided by some motion planner (e.g. [11]).

Linking a translation and a rotation forces the robot to stop. At the stop point, both linear and angular velocities $v$ and $\omega$ are zero. The trajectory appears as a sequence such as (Stop--TRANSLATION--Stop--ROTATION--Stop--TRANSLATION--...--Stop)$^4$. The goal of the smoothing is to remove the stop points, while maintaining the continuity of the velocities $v$ and $\omega$, and controlling the deviation of the trajectory after smoothing from the initial one. At any point of the resulting trajectory, either $\omega$ or $v$ are non zero.

4.1 The canonical motion [T-s-R-s-T]
To this end, we introduce canonical primitives for smoothing sub-sequences of trajectory such as [TRANSLATION--Stop--ROTATION--Stop--TRANSLATION] or [T-s-R-s-T]). We want to produce a smooth trajectory symmetrical with respect to the bisector of the lines supporting consecutive translations separated by a rotation.

Notations : We note $S_1$ the first straight line segment and $S_2$ the second one (see figure 5). $P$ is the intersection point of $S_1$ and $S_2$. $\alpha$ is the measure of the angle of the two lines supporting $S_1$ and $S_2$ ($0 < \alpha < \pi$ the case $\alpha < 0$ can be deduced by symmetry). Moreover we assume that the direction of motion on $S_1$ is positive (i.e., the robot moves forward). The case of a backward motion can also be deduced by symmetry. We note $\sigma$ the direction of motion on $S_2$ : $\sigma = 1$ if the robot moves forward and $\sigma = -1$ otherwise. Finally, $\delta$ is the direction of rotation at $P$ : $\delta = 1$ if the robot rotates counterclockwise and $\delta = -1$ otherwise.

The inputs of the problem are an elementary sequence [T-s-R-s-T] and a safety parameter defined for collision avoidance, i.e.:

- two straight line segments $S_1$ and $S_2$, their length $e_i$, intersection point $P$ and angle $\alpha$.

$^4$or: (Stop--ROTATION--Stop--TRANSLATION--...--Stop)
- \( \sigma \) the imposed direction of motion on \( S_2 \).
- \( \delta \) the imposed direction of rotation at \( P \).
- a positive number \( \epsilon \) measuring the maximum deviation between the reference trajectory \((S_1, S_2)\), and the resulting smoothed trajectory.

\[ z_0 = e - \epsilon \left( \cot \frac{\alpha}{2} + CF \left( \sqrt{\frac{s^2}{r}} \right) \right) \]

\[ k_e = \frac{SF \left( \sqrt{\frac{s^2}{r}} \right)}{\epsilon} \]

\([x_0, 0, 0]\) is the initial configuration of the clothoid.
Now, we have to compute \( k_e \) and \( x_0 \):

\[ x_0 = e - \epsilon \left( \cot \frac{\alpha}{2} + CF \left( \sqrt{\frac{s^2}{r}} \right) \right) \]

\[ y_0 = \frac{x_0 \cdot SF \left( \sqrt{\frac{s^2}{r}} \right)}{\epsilon} \]

The first clothoid arc is then computed. The second one is deduced by symmetry. Figure 6 shows the resulting smoothed trajectory.

4.2 \( \sigma = 1 \) and \( \delta = 1 \): the turn

This is a classical case. The smoothing primitive consists of two arcs of clothoid that meet at a configuration \( c \) such that:

\[ \begin{align*}
  x_c &= e + \epsilon \cos \frac{\alpha}{2} \\
  y_c &= -\epsilon \sin \frac{\alpha}{2} \\
  \theta_c &= \pm \frac{\alpha}{2}
\end{align*} \]

We want to compute an arc of clothoid linking the \( x \)-axis to \( c \). From equation (4) we can substitute \( t \) by \( \sqrt{\frac{2s_e}{\epsilon}} \) in equations (2) and (3):

\[ x_e = \sqrt{\frac{2s_e}{\epsilon}} CF \left( \sqrt{\frac{s^2}{r}} \right) + x_0 \]

\[ y_e = \frac{x_e \cdot SF \left( \sqrt{\frac{s^2}{r}} \right)}{\epsilon} \]

4.3 \( \sigma = 1 \) and \( \delta = -1 \): the loop

This is a classical case. The smoothing primitive consists of two clothoid arcs (see figure 7):

\[ \begin{align*}
  x_c &= e + \epsilon \cos \frac{\alpha}{2} \\
  y_c &= -\epsilon \sin \frac{\alpha}{2} \\
  \theta_c &= \pm \frac{\alpha}{2}
\end{align*} \]

4.4 \( \sigma = -1 \) and \( \delta = 1 \): the internal cusp

The case of cusps is more complicated. The primitive of smoothing consists of two arcs of clothoid linked
by two arcs of anticlothoid, symmetrical with respect to the bisector of $S_1$ and $S_2$.

Consider the configuration $c$ lying on the bisector and verifying:

$$
\begin{align*}
  x_c &= e - \epsilon \cot \frac{\alpha}{2} \\
  y_c &= \epsilon \\
  \theta_c &= \frac{2\pi - \alpha}{2}
\end{align*}
$$

We build a sequence consisting of an arc of a clothoid and an arc of an anticlothoid starting from the $x$-axis to $c$.

Let $\hat{c}$ be the configuration at the switching between the clothoid and the anticlothoid. Being the meeting point of a clothoid and an anticlothoid, $\hat{c}$ verifies equation (9) and:

$$
\begin{align*}
  x_{\hat{c}} &= \sqrt{\frac{x}{x_0}} CF(\sqrt{\frac{x}{x_0}}) + x_0 \\
  y_{\hat{c}} &= \sqrt{\frac{x}{x_0}} SF(\sqrt{\frac{x}{x_0}}) \\
  \theta_{\hat{c}} &= \frac{2\pi - \alpha}{2}
\end{align*}
$$

Now, $c$ is the endpoint of an arc of an anticlothoid whose initial configuration is $\hat{c}$. From the computations done in Section 2.4 (equation 10) and in the appendix (equations (11) and (12)):

$$
\begin{align*}
  x_c &= k_a(\cos \theta_c - \cos \theta_0 + (\theta_0 - \theta_c) \sin \theta_0) + x_{\hat{c}} \\
  y_c &= k_a(\sin \theta_c - \sin \theta_0 - (\theta_0 - \theta_c) \cos \theta_0) + y_{\hat{c}} \\
  \theta_c &= \frac{\alpha}{4k_a} + \sqrt{\frac{2}{4k_a^2}}
\end{align*}
$$

Therefore the indeterminates $x_0$, $k_a$ and $k_c$ verify the following system:

$$
\begin{align*}
  e - \epsilon \cot \frac{\alpha}{2} &= k_a(\cos \theta_c - \cos \theta_0 + \sqrt{\frac{d}{4k_a^2}} \sin \frac{\alpha}{4k_a}) + \sqrt{\frac{d}{4k_a^2} CF(\sqrt{\frac{d}{4k_a^2}}) + x_0} \\
  \epsilon &= k_a(\sin \theta_c - \sin \frac{\alpha}{4k_a} - \sqrt{\frac{2}{4k_a^2} \cos \frac{\alpha}{4k_a}}) + \sqrt{\frac{d}{4k_a^2} SF(\sqrt{\frac{d}{4k_a^2}}) + x_0} \\
  2\pi - \alpha &= \frac{\alpha}{4k_a} + \sqrt{\frac{2}{4k_a^2}}
\end{align*}
$$

From the third equation we can compute $k_a$ as a function of $k_c$. By replacing $k_c$ in the second equation, we obtain an equation with the only indeterminate $k_a$. We solve this new equation numerically. Finally $x_0$ is given by the first equation.

Figure 8 shows an example of trajectory composed of this primitive.

4.5 $\sigma = -1$ and $\delta = -1$ : the external cusp

Here, the primitive is exactly the same as the previous one, except that we consider the intermediate following configuration $c$ (Figure 9):

$$
\begin{align*}
  x_c &= e + \epsilon \cos \frac{\alpha}{2} \\
  y_c &= -\epsilon \sin \frac{\alpha}{2} \\
  \theta_c &= \frac{\alpha}{2}
\end{align*}
$$

The smoothed trajectory lies on the other side of the reference trajectory $(S_1, S_2)$ (Figure 9).

4.6 Geometrical property for collision avoidance

All the above trajectories are clearly produced by smooth controls. According to the kinematic properties of the motions along clothoids and anticlothoids, the velocities of the wheels of the mobile robot are never zero simultaneously.

We have now to check that all the four trajectories above verify the geometric constraints imposed by the inputs.

Property : The four primitives described above are trajectories $\gamma$ lying at a distance less than $\epsilon$ from $S_1$ and $S_2$.

Proof : The proof comes from the choice of the intermediate configuration $c$ lying on the bisector of segments $S_1$ and $S_2$, and from the convexity of the arcs of clothoids and anticlothoids used to construct $\gamma$ in each case.

Consider the "tube" consisting of all the points whose distance to $S_1$ and $S_2$ is less than $\epsilon$. It is bounded by:

- two segments intersecting at point $C_1$ with coordinates $(e - \epsilon \cot \frac{\alpha}{2}, \epsilon)$, and
- two segments and an arc of a circle intersecting the bisector at point $C_2$ with coordinates $(e + \epsilon \cos \frac{\alpha}{2}, -\epsilon \sin \frac{\alpha}{2})$.

In the cases of the internal cusp and the turn, the intermediate configuration lies at point $C_1$. From convexity arguments, the portion of $\gamma$ from the origin to $c$ lies under the line $(OC_1)$ and over the $x$-axis, and therefore inside the tube (figure 10).
Figure 10: The trajectory $\gamma$ lies inside the tube of width $\epsilon$.

Similarly, in the cases of the loop and the external cusp, the portion of $\gamma$ from the origin to $c$ lies over the line $(OC_2)$ and under the $x$-axis. This suffices to conclude for the case of the external cusp. For the loop, we can check that the curvature of $\gamma$ at configuration $c$ is less than $\epsilon$. Thus, the first portion lies inside the convex part of the domain bounded by the arc of circle centered at $P$ and of radius $\epsilon$. Therefore, it lies inside the tube. By symmetry with respect to the bisector, the second portion of $\gamma$ also lies inside the tube. $\Box$

Remarks:
1. Note that not only $\gamma$ is close to the segments in the cartesian space, but also the corresponding path in the configuration space is also as close as we want to the initial path supported by the two segments and the rotation at $P$: indeed the angular variation of the robot's orientation moving along $\gamma$ is exactly the same as the initial angular variation along $(S_1, P, S_2)$. This point is very important in constrained space when collision risks are due to the rotations.

2. In situations geometrically very constrained, the length $\epsilon$ of $S_1$ may be too small and the above computations may lead to $x_0 < 0$. In this case, we fix $x_0$ to zero. $\epsilon$ becomes an indeterminate which can be computed. The final trajectory will then be closer to the segments than the initial geometric constraints of the inputs.

4.7 Comments
All the above computations are founded on the geometric properties of the clothoids and the anticlothoids. Recall that these curves are fully characterized by the parameters $k_1$, $k_2$, and $k_3$.

From the kinematic viewpoint, $k_2$ and $k_3$ can be expressed as functions of the velocities $v_0$ and $\omega_1$, and the constant acceleration $a$. If the value of $a$ is not fixed, the trajectory being given, we have one degree of freedom for computing the motion along the trajectory.

5 Conclusion
The presented primitives can be linked together to smooth a reference trajectory composed of oriented broken lines, with constraints on the deviation from the reference trajectory. Producing an optimal smooth path from the input sequence of segments by the means of an algorithm is still an unsolved problem that we are investigating. The objective is to produce a trajectory like the one shown in figure 11. In this example the cusps and the loop were imposed manually to the system. The swept space during motion is shown in figure 12.

Figure 11: A sequence of primitives linking a turn, an external cusp, an internal cusp, and a loop.

Figure 12: Space swept by the robot during execution of the trajectory of figure 11.

We addressed the case of a sequence Translation-Rotation-Translation. The same approach would have been also possible for the case Rotation-Translation-Rotation. The produced smooth trajectory would pass through the vertices of the reference trajectory.

Finally, we recall that even if the smoothing was produced by "bang-bang" commands, this does not guarantee the optimality of the trajectory. Characterization of the optimal trajectories is still an open problem.

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Appendix: Parametric equations of the anticlothoid

In this appendix, we present the computation of the configuration $c(t)$ of a mobile robot moving along an anticlothoid $\gamma$ from a start configuration $c_0 = (x_0, y_0, \theta_0)$ with initial linear and angular velocities $v_0$ and $\omega_0$.

The coordinates of $c(t)$ are:

\[
\begin{align*}
    x(t) &= \int_0^t v(u) \cos \theta(u) \, du + x_0 \\
    y(t) &= \int_0^t v(u) \sin \theta(u) \, du + y_0 \\
    \theta(t) &= \omega_0 t + \theta_0
\end{align*}
\]

From the definition of anticlothoids (see Section 2.3), we have $\omega(t) = \omega_0$ and $v(t) = at + v_0$. Moreover, the characteristic parameter of the anticlothoid is $k_a = \frac{a}{\omega_0^2}$. Thus:

\[
\begin{align*}
    x(t) &= k_a (\cos \theta - \cos \theta_0 + (\theta - \theta_0 + \frac{\omega_0 t}{a}) \sin \theta - \frac{\omega_0 t}{a} \sin \theta_0) + x_0 \\
    y(t) &= k_a (\sin \theta - \sin \theta_0 - (\theta - \theta_0 + \frac{\omega_0 t}{a}) \cos \theta + \frac{\omega_0 t}{a} \cos \theta_0) + y_0 \\
    \theta(t) &= \omega_0 t + \theta_0
\end{align*}
\]

When $v_0 < 0$, there is a point $\gamma(t)$ such that $v(t) = 0$. This is a cusp. The corresponding configuration $c$ verifies:

\[
\begin{align*}
    x_0 &= k_a (\cos \theta - \cos \theta_0 - (\theta - \theta_0) \sin \theta_0) + x_0 \\
    y_0 &= k_a (\sin \theta - \sin \theta_0 - (\theta - \theta_0) \cos \theta_0) + y_0 \\
    \theta_0 &= -\frac{v_0 \omega_0}{a} + \theta_0
\end{align*}
\]

References


