On the Formation Rejoin Problem

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Flight lead frame

- **Flight lead frame** is parametrized at time $t \in \mathbb{R}$ by its position and orientation, $x_d(t) \in \mathbb{R}^3$ and $R_d(t) \in SO(3)$, in the spatial frame.

- **Admissible lead trajectories** restricted to those such that the velocity, acceleration and angular velocity, $\dot{x}_d(t)$, $\ddot{x}_d(t)$ and $\omega_d(t)$ are bounded, with $\omega_d(t)$ satisfying

$$\dot{R}_d(t) = R_d(t) \hat{\omega}_d(t).$$  \hfill (1)

- **Hat operator**: matrix representation of the cross product

$$\hat{\omega} v = \omega \times v, \quad \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- **Rotation matrix** $R_d(t)$ maps local vectors to the spatial frame

$$y_{\text{spatial}} = R_d(t)y_{\text{local}}.$$
Relative Dynamics

- **Flight lead frame** wingman’s position, velocity and acceleration
  \[
  p = R_d^T(t) [x - x_d(t)] \\
  v = R_d^T(t) [\dot{x} - \dot{x}_d(t)] \\
  a = R_d^T(t) [\ddot{x} - \ddot{x}_d(t)]
  \]

- **Local dynamics**: state \((p, v)\), input \(a\)
  \[
  \dot{p} = v - \hat{\omega}_d(t) p \\
  \dot{v} = a - \hat{\omega}_d(t) v
  \]
Energy Considerations

We expect the energy of the system to be unaffected by $\hat{\omega}_d(t)$ since it does no work.

- Let $T = \frac{1}{2} v^T v$ be some sort of kinetic energy, then
  $$\dot{T} = -v^T \hat{\omega}_d(t) v + v^T a = v^T a$$

- Let the potential be a pure quadratic, $U = \frac{1}{2} k_p p^T p$, $k_p > 0$, then the total energy is
  $$E = \frac{1}{2} k_p p^T p + \frac{1}{2} v^T v$$
  and the derivative is
  $$\dot{E} = k_p p^T (-\hat{\omega}_d(t) p + v) + v^T a = k_p p^T v + v^T a$$

- What property does the kinetic and potential energy posses that allows the energy to evolve independent of $\hat{\omega}_d(t)$?
We expect the energy of the system to be unaffected by \( \hat{\omega}_d(t) \) since it does no work.

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  \dot{T} = -\mathbf{v}^T \hat{\omega}_d(t) \mathbf{v} + \mathbf{v}^T \mathbf{a} = \mathbf{v}^T \mathbf{a}
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  \]

- What property does the kinetic and potential energy possess that allows the energy to evolve independent of \( \hat{\omega}_d(t) \)?

Rotational Invariance
Rotational Invariance

**Definition**

$U : \mathbb{R}^3 \rightarrow \mathbb{R}$ is rotationally invariant if

$$U(Rp) = U(p) \text{ for all } p \in \mathbb{R}^3 \text{ and } R \in SO(3).$$
Rotational Invariance

**Lemma**

\[ U : \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is rotationally invariant if and only if} \]

\[ DU(p) \cdot \hat{\omega} p = 0, \quad \forall p, \omega \in \mathbb{R}^3. \]

**Proof.**

\((\Rightarrow\)\) Let \( p, \omega \in \mathbb{R}^3 \) be arbitrary and set \( R(t) = e^{t\hat{\omega}} \in SO(3), \quad t \in \mathbb{R}. \)

Then, since \( U(R(t)p) = U(p), \quad t \in \mathbb{R}, \)

by rotational invariance, we see that

\[
0 = \frac{d}{dt} \{ U(R(t)p) \} = DU(R(t)p) \cdot \hat{\omega} R(t)p
\]

\(\Leftarrow\) Let \( \tilde{R} \in SO(3) \) and \( p \in \mathbb{R}^3 \) be arbitrary, choose \( \omega \in \mathbb{R}^3 \) such that \( \tilde{R} = e^{\hat{\omega}} \) and define \( R(t) = e^{t\hat{\omega}}, \quad t \in \mathbb{R}, \) so that \( R(1) = \tilde{R}. \)

Then

\[
U(R(t)p) = U(p) + \int_0^1 DU(R(\tau)p) \cdot \hat{\omega} R(\tau)p \, d\tau = F(p)
\]
Geometry

Level sets of a rotationally invariant function are spherical.

Corollary

∇U(p) || p and DU(p) · p > 0 (away from p = 0) for an increasing rotationally invariant function U(p).

Summary: If the kinetic and potential energy are rotationally invariant the total energy

\[ E(p, v) = T(v) + U(p) \]

will evolve independently of \( \omega_d(t) \) according to

\[ \dot{E} = DU(p) \cdot v + DT(v) \cdot a \]
Level sets of a rotationally invariant function are spherical.

**Corollary**

\[ \nabla U(p) \parallel p \text{ and } DU(p) \cdot p > 0 \text{ (away from } p = 0) \text{ for an increasing rotationally invariant function } U(p). \]

**Summary:** If the kinetic and potential energy are rotationally invariant the total energy

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will evolve independently of \( \omega_d(t) \) according to

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general rotational invariance: \( E(p, v) = E(Rp, Rv) \)
Feedback Stabilization

Open loop structure looks like a double integrator, suggesting the use of a PD controller to stabilize the system.

- For the moment let’s entertain the controller:

\[ a = -K_p \dot{p} - K_v v \]

with \( K_p \) and \( K_v \) symmetric and positive definite.

- Provides exponential stability for \( \omega_d(t) = 0 \). What about a general constant \( \omega_d \)?
Consider a constant turning maneuver in the XY plane with

- $\omega_d = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$
- $K_v = I$
- $K_p = \text{diag}(\begin{bmatrix} a & b & 1 \end{bmatrix})$ with $a, b > 0$

For initial conditions in horizontal plane, the system evolves within that plane as $\dot{x} = Ax$ with

$$A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
-a & 0 & -1 & 1 \\
0 & -b & -1 & -1 \\
\end{bmatrix}$$

giving a characteristic polynomial of

$$\chi(s) = s^4 + 2s^3 + (a + b + 3)s^2 + (a + b + 2)s + ab - a - b + 2.$$  

The system is unstable if $a + b > ab + 2$, for instance, $a = 4$ and $b = 1/2$. 
General Stabilizing Feedback

Why doesn’t the general $Kp > 0$ work?
- the associated potential $U(p) = \frac{1}{2}p^T K_p p$ (for proportional feedback $-K_p p = -\nabla U(p)$) is not rotationally invariant.

One solution:
- Use a rotationally invariant potential $U(p)$ that is positive definite and a control law of the form
  \[ a = -\nabla U(p) - \bar{k}_v(v) \]  
  (2)

Then the energy
  \[ E(p, v) = U(p) + \frac{1}{2}v^T v \]  
  (3)

evolves according to
  \[ \dot{E} = -v^T \bar{k}_v(v) . \]  
  (4)

If $\bar{k}(v) \equiv 0$ then the energy is conserved, and if $v^T \bar{k}_v(v) > 0$ for nonzero $v$ then the energy will dissipate over time.
Exponential Stability in LTI case

Suppose $\omega_d(t) = \omega_d$ is constant.

If

- Potential: $U(p) = \frac{1}{2} k_p p^T p$, $k_p > 0$
- Feedback:

$$a = -k_p p - K_v v,$$

where $K_v$ is positive definite (and usually symmetric).

Then the energy evolves according to $\dot{E} = -v^T K_v v \leq 0$ and the closed loop system is exponentially stable.

Pf. LaSalle’s invariance principle
Exponential Stability of LTV

We wish to augment the energy in order to convert it from a weak Lyapunov function to a strict one.

- Extend the notion of rotational invariance to functions with two arguments.

**Definition**

\[ W : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is rotationally invariant if and only if} \]

\[ W(Rp, Rv) = W(p, v) \text{ for all } p, v \in \mathbb{R}^3 \text{ and all } R \in SO(3). \]

**Lemma**

\[ W(p, v) \text{ is rotationally invariant if and only if} \]

\[ D_1 W(p, v) \cdot \hat{\omega} p + D_2 W(p, v) \cdot \hat{\omega} v = 0 \text{ for all } p, v, \omega \in \mathbb{R}. \]
Strict Lyapunov Function

Introduce the rotationally invariant cross term \( W(p, v) = p^T v \) with \( \omega_d(t) \) free derivative

\[
\dot{W}(p, v, a) = v^T v + p^T a.
\]

Lyapunov function

\[
V(p, v) = E(p, v) + \epsilon W(p, v) = \frac{1}{2}k_p p^T p + \frac{1}{2}v^T v + \epsilon p^T v
\]

- time independent
- quadratic form \( V(p, v) = \frac{1}{2}x^T P x \), \( x = [p; v] \).
- positive definite provided that

\[
P = \begin{bmatrix} k_p & \epsilon \\ \epsilon & I \end{bmatrix} > 0
\]

which occurs when \( \epsilon^2 < k_p \).
Strict Lyapunov Function

Derivative of the Lyapunov function

\[
\dot{V}(p, v) = \dot{E}(p, v) + \epsilon \dot{W}(p, v) = v^T K_v v - \epsilon p^T K_v v^T - \epsilon k_p p^T p + \epsilon v^T v
\]

- time independent
- quadratic form \( \dot{V}(p, v) = -\frac{1}{2} x^T Q x \)
- for general (possibly non-symmetric) \( K_v \), \( K_v^s = \frac{K_v + K_v^T}{2} > 0 \)

\[
Q = \begin{bmatrix}
2\epsilon k_p I & \epsilon K_v \\
\epsilon K_v^T & 2(K_v^s - \epsilon I)
\end{bmatrix} > 0
\]

when

\[
\epsilon < \lambda_{\min}(4k_p K_v^s, (K_v^T K_v + 4k_p I)).
\]

- for symmetric \( K_v \)

\[
\epsilon < 1/\lambda_{\max}(K_v^{-1} + K_v/4k_p).
\]
Strict Lyapunov Function

Proposition

For each $k_p > 0$ and $K_v \in \mathbb{R}^{3 \times 3}$ with $K_s = (K_v + K_v^T)/2 > 0$, the linear feedback (5) exponentially stabilizes the maneuvering system (3) where lead maneuvers with a locally bounded angular velocity $\omega_d(\cdot)$.

- local boundedness condition ensures uniqueness.
- a uniform bound on $\omega_d(\cdot)$ is not required
- $\omega_d(\cdot)$ need not be differentiable.
If the LTI system $\dot{x} = Ax$ with quadratic Lyapunov function $V(x) = x^T Px$, $P^T = P > 0$ is stable, then there exist an $\alpha > 0$ such that

$$A^T P + PA + 2\alpha P \leq 0$$

and $\|x(t)\| \leq \kappa(P)^{\frac{1}{2}} \|x(0)\| e^{-\alpha t}$

The actual decay rate $\alpha = -\max_k \text{real} \lambda_k(A)$ can be determined by solving the generalized eigenvalue problem in $P$ and $\alpha$

$$\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subj to} & \quad P > 0 \\
& \quad A^T P + PA + 2\alpha P \leq 0
\end{align*}$$

which provides a Lyapunov function proving that decay rate.
Decay Rate Estimate

- LTV case, let the closed loop system be described by
  \[ \dot{x} = (A + A_{\omega_d}(t)) x, \]
  \[ A = \begin{bmatrix} 0 & I \\ -k_p I & -K_v \end{bmatrix}, \quad A_{\omega_d}(t) = \begin{bmatrix} -\hat{\omega}_d(t) & 0 \\ 0 & -\hat{\omega}_d(t) \end{bmatrix} \]

- Using the Lyapunov function \( V(p, v) = \frac{1}{2} x^T P(\epsilon) x, \ x = [p; v] \) with
  \[ P(\epsilon) = \begin{bmatrix} k_p I & \epsilon I \\ \epsilon I & I \end{bmatrix} \]

then by rotational invariance, \( A^T_{\omega_d}(t) P = PA_{\omega_d}(t) = 0_{6 \times 6} \)

Estimated Decay Rate of LTV

\[
\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subj to} & \quad P(\epsilon) > 0 \\
& \quad \epsilon > 0 \\
& \quad A^T P(\epsilon) + P(\epsilon) A + 2\alpha P(\epsilon) \leq 0
\end{align*}
\]
Analytic decay rate for scalar $K_v$

Letting $K_v = k_v I$ we can consider the scalar system $x = (p, v)^T \in \mathbb{R}^2$ so that the corresponding quadratic form matrices are

$$P(\epsilon) = \begin{bmatrix} k_p & \epsilon \\ \epsilon & 1 \end{bmatrix} \quad \text{and} \quad Q(\epsilon) = \begin{bmatrix} 2\epsilon k_p & \epsilon k_v \\ \epsilon k_v & 2(k_v - \epsilon) \end{bmatrix}$$

and the decay rate, as a function of $\epsilon$ feasible, is given by

$$\alpha(\epsilon) = \frac{1}{2} \min_{\|y\|=1} \frac{y^T Q(\epsilon) y}{y^T P(\epsilon) y}.$$ 

Letting $R(\epsilon)$ be the Cholesky factor of $P(\epsilon) > 0$ ($P(\epsilon) = R^T(\epsilon) R(\epsilon)$), we see that

$$\alpha(\epsilon) = \frac{1}{2} \lambda_{\min}(R^{-T}(\epsilon) Q(\epsilon) R^{-1}(\epsilon))$$

where

$$R^{-T}(\epsilon) Q(\epsilon) R^{-1}(\epsilon) = \begin{bmatrix} 2\epsilon & \epsilon \frac{k_v - 2\epsilon}{\sqrt{k_p - \epsilon^2}} \\ \epsilon \frac{k_v - 2\epsilon}{\sqrt{k_p - \epsilon^2}} & 2(k_v - \epsilon) \end{bmatrix}$$
Equations of Motion

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Example: figure eight rejoin

\[
\begin{bmatrix}
2\epsilon & \epsilon \frac{k_v-2\epsilon}{\sqrt{k_p-\epsilon^2}} \\
\epsilon \frac{k_v-2\epsilon}{\sqrt{k_p-\epsilon^2}} & 2(k_v - \epsilon)
\end{bmatrix}
\]

characteristic polynomial:

\[\lambda^2 - 2k_v \lambda + \epsilon \frac{4k_p k_v - \epsilon(4k_p + k_v^2)}{k_p - \epsilon^2}\]

re-parametrize:

- \(k_p = \omega_n^2\), \(\omega_n > 0\)
- \(k_v = 2\zeta \omega_n\), \(\zeta \in [0, 1)\)

\(\epsilon\) constraints:

\[\epsilon < k_v \min \left\{ \frac{1}{2\zeta}, 1, \frac{1}{1+\zeta^2} \right\} = \frac{k_v}{1+\zeta^2}\]

parametrize feasible \(\epsilon\):

\[\epsilon = \delta \frac{k_v}{1+\zeta^2}\] with \(\delta \in (0, 1)\)

discriminant:

\[\frac{(\zeta^2 - 2\delta + 1)^2}{\zeta^4 + (2 - 4\delta^2)\zeta^2 + 1} k_v^2\]

discriminant is zero when \(\delta = (1 + \zeta^2)/2\) at which point the minimum eigenvalue is \(k_v\), giving a decay rate of \(\alpha = k_v/2\) for the LTV system
Note that, when $K_v$ is taken to be scalar, then the control action acceleration is a simple linear combination of the position and velocity vectors.

This means that the feedback is essentially coordinate free, meaning that $p$, $v$, and $a$ can be expressed in any coordinate system as long as the same is used for all three.

In particular, the maneuvering vehicle pilot can determine his control action within any convenient frame.
Nonlinear Feedback

- **Motivation**: Address the saturation limits of the achievable acceleration (and velocity).
- **Framework**: Already exists!

\[ a = -\nabla U(p) - \bar{k}_v(v) \quad (\diamond 2) \]

\[ E(p, v) = U(p) + \frac{1}{2}v^Tv \quad (\diamond 3) \]

\[ \dot{E} = -v^T\bar{k}_v(v) \quad (\diamond 4) \]

- **Question**: How do we choose \( U(p) \) and \( \bar{k}_v(v) \) to ensure stability and attractiveness of the nonlinear system?
Conditions on $U(p)$

- rotationally invariant
- positive definite ($U(0) = 0$ and $U(p) > 0$, $p \neq 0$) locally
- radially unbounded (for global results)

Nice to haves:

- $\nabla U(p) \neq 0$ for all $p \neq 0$ (potential energy grows)
- quadratic in a neighborhood of the origin

Conditions on $\bar{k}_v(v)$

- $v^T \bar{k}_v(v) > 0$ whenever $v \neq 0$.
- $\bar{k}_v(0) = 0$.

Nice to have:

- $v^T \bar{k}_v(v)$ is rotationally invariant
Theorem

Suppose that

- $U: \mathbb{R}^3 \to \mathbb{R}$ is a $C^2$ rotationally invariant, positive definite and radially unbounded function such that $\nabla U(p) \neq 0$ for all $p \neq 0$,
- $\bar{k}_v : \mathbb{R}^3 \to \mathbb{R}^3$ is a $C^1$ mapping such that $\bar{k}_v(0) = 0$ and $v^T \bar{k}_v(v) > 0$ for all $v \neq 0$,
- $\omega_d : \mathbb{R} \to \mathbb{R}^3$ is bounded.

Then, the closed loop system is uniformly globally asymptotically stable. If the Hessian $D^2 U(0)$ and the Jacobian $D\bar{k}_v(0)$ are both full rank, then the closed loop system is locally exponentially stable.
Proof.

Local exponential stability:
- apply Proposition 7 to the linearization at zero
- local quadratic approximation of $U(p)$.

Nonlinear part:
- Lyapunov function $V(x) = E(p, v)$ (uniformly globally stable)
- let $E = \{ x = (p, v) : v = 0 \}$ be the $\dot{V} = 0$ set
- set $W(x) = p^T v$ and $\dot{W}(x) = -p^T \nabla U(p) + v^T v - p^T \tilde{k}_v(v)$
- on the set $E \setminus \{0\}$, $\dot{W}(p, 0) = -p^T \nabla U(p) < 0$ so that we can’t stay near $E$ and away from the origin for very long.

These properties of $V$ and $W$ and the fact that $\omega_d(\cdot)$ is bounded allow us to conclude, by Matrosov’s Theorem, that the origin is a uniformly globally asymptotically stable equilibrium for the closed loop system.
Example: Figure Eight Rejoin

**Maneuver Parameters:**
- lead velocity: 10 m/s
- max lateral acc.: 7.2 m/s² (0.73 g’s)
- lead in coordinated flight (accelerations constrained to normal and axial directions along)

**Figure:** Rejoin to a *figure eight* in the N-E spatial frame.
Potential Function:

\[ U(p) = \frac{a_m^2}{k_p} \log \cosh \frac{k_p}{a_m} \|p\| \]

Position Feedback:

\[ \nabla U(p) = \left( \frac{a_m}{\|p\|} \tanh \frac{k_p}{a_m} \|p\| \right) p \]

Velocity Feedback: analogous to position feedback

**Figure:** Rejoin to a figure eight maneuver: flight lead angular velocity \( \omega_d(t) \) (top) together with the resulting relative position (mid) and velocity (bottom) trajectories.

**Figure:** Acceleration feedback component magnitudes.

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**Equations of Motion**

**Energy Considerations**

**Feedback Stabilization**

Example: figure eight rejoin
Figure: Lyapunov and auxiliary function values for figure eight rejoin maneuver.

- auxiliary function $W$ ensures that the system trajectory does not stay close to the $\dot{V} = 0$ for long when there is energy to burn